THE CYLINDRICAL ANTENNA; CURRENT AND IMPEDANCE*

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1. Introduction. The definition and the determination of the impedance of a symmetrical, center-driven antenna of small, circular cross section involves three major problems. These are first the theoretical analysis including the formulation of boundary conditions; second the apparatus and the technique of experimental measurement; and third the coordination of experiment with theory. Of these problems only the first is the subject of this paper; the last two are considered in detail elsewhere.[1] The present discussion is concerned specifically with an analytical improvement in the solution of the theoretical problem.

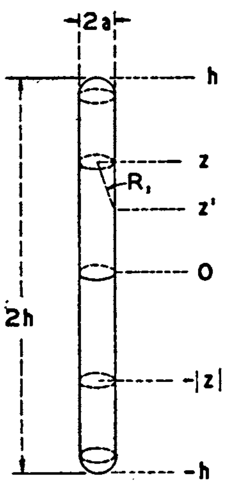

Fig. 1. Cylindrical antenna with hemispherical ends.

The boundary and driving conditions in this analysis are the same as implied in earlier analyses,[2,3] and the same integral equation is obtained. However, the present paper introduces a new approach to the solution of Hallén's integral equation in that it replaces a function arbitrarily chosen for reasons of mathematical convenience in the approximate evaluation of the equation by a function *actually fitted to the true distribution of current.* As a consequence, new parameters are introduced to replace those used by Hallén (or equally those used by Gray[12]) in the successive approximations, and as would be expected the resulting development shows a more rapid convergence, in so far as this is indicated by a relatively small difference between first and second order solutions.

The antenna actually analyzed is a theoretical one in the sense that no exact experimental analogue can be constructed. Its properties are summarized as follows:

(1) The antenna is a highly conducting cylinder of small radius a extending unbroken from $z=-h$ to $z=+h$ as shown in Fig. 1. Postulated inequalities are

$$\beta a \ll 1, \qquad a \ll h, \tag{1}$$

where $\beta=\omega/c$ is the phase constant and $c=3\times 10^8$ m/sec.

(2) The ends of the antenna at $z=\pm h$ contribute nothing to the electrical problem so that it is correct to write

$$I_h = 0 \quad \text{at} \quad z = \pm\, h. \tag{2}$$

(3) The antenna is center-driven by a slice generator consisting of a disk of neg-

* Received Aug. 6, 1945.

[1] R. King and D. D. King. J. Appl. Phys. 16, 445 (1945).

[2] E. Hallén, Nova Acta, Royal Soc. Sciences, Upsala 11, 1 (1938).

[3] R. King and C. W. Harrison, Jr., Proc. I.R.E. 31, 548 (1943).

ligible thickness at the center, $z=0$, which is in all respects like any other piece of the antenna except that a scalar potential difference

$$V_0^e = \lim_{z\to 0} (\phi_{+z} - \phi_{-z}) = \phi_{+0} - \phi_{-0} \tag{3}$$

is maintained between its faces.

(4) All other conductors and all dielectrics are sufficiently far away so that their individual effects are indistinguishable from the composite effect of the universe as a whole. If R is the distance from the center of the antenna to the nearest point on any other conductor or on a dielectric the following inequalities must be satisfied

$$\beta R \gg 1; \qquad R \gg h. \tag{4}$$

The degree in which this theoretical antenna can be realized physically is summarized briefly below. Details are found elsewhere.[1,4,5,6,7]

(a) A metal wire or rod can be constructed to satisfy completely the properties assumed in conjunction with (1).

(b) If a solid cylinder with flat ends or a hollow cylinder is used (2) is not exactly true. A small current exists at the ends to charge the sharp edges, the end surfaces, or the inner surfaces of a tube near the ends. This leads to an error in h of the order of magnitude of a, and a consequent hidden shift in the theoretical impedance curves. For particular values of h near anti-resonance large errors in impedance are involved. A solid cylinder of length $2h$ along the axis with hemispherical ends as shown in Fig. 1 is a satisfactory physical approximation that satisfies (1) and (2).

(c) It is physically impossible to provide a slice generator. At very low frequencies a two-wire drive is satisfactory to approximate (3) but in this case (4) can not be satisfied. At high frequencies where (4) is readily satisfied a two-wire drive involves adjacent end surfaces and a gap in the antenna which are not taken into account in the theory. The effects of gap and end surfaces are compensating and may be taken into account roughly in comparing theoretical and experimental results by including a lumped capacitance in parallel with the experimentally measured impedances if the gap is large and a similar capacitance in parallel with the theoretical impedances if the gap is very narrow and the adjacent end surfaces of the antenna are very close together.[1] A vertical antenna of length h over a conducting plane, driven from a coaxial line, may be a good approximation of a slice generator, but the unavailability of an infinite, perfectly conducting plane leads to other difficulties.

(d) The condition for the far zone (4) can not be fulfilled at low radio frequencies (where accurate measurements can be made easily) because it is not possible to get far enough away from the earth. At high frequencies where this is possible, accurate measurements are difficult and the dimensions of the antenna and its driving structure become undesirably small.[1]

The analysis of the theoretical antenna subject to the conditions (1) to (4) discussed above, can be reduced to one-dimensional form involving the total current I_z if it is assumed that the cross-sectional distribution of the density of current is inde-

[4] L. Brillouin, Quart. Appl. Math. **1**, 201 (1943).
[5] L. Brillouin, El. Communication **22**, 11 (1944).
[6] S. A. Schelkunoff, J. Appl. Phys. **15**, 54 (1944).
[7] R. King and C. W. Harrison, Jr., J. Appl. Phys. **15**, 170 (1944).

pendent of the axial distribution. In effect, this means that the cross-sectional distribution and the internal impedance per unit length may be obtained from the analysis of an infinitely long cylinder. This is an excellent approximation subject to (1). At high frequencies the internal impedance per unit length is given by

$$z^i = \frac{1}{2\pi a}\sqrt{\frac{\omega\mu\Pi}{2\sigma}}\,(1 + j) \tag{5}$$

with σ the conductivity in mhos per meter, μ the relative permeability, a the radius in meters, and $\Pi = 4\pi \times 10^{-7}$ henry per meter. Subject to this assumption in addition to (1)–(4) the vector potential at any point z on the cylindrical surface of the antenna due to the axial current in the entire antenna is given by [3,8,9]

$$A_z = \frac{\Pi}{4\pi}\int_{-h}^{h} I_z' R_1^{-1} e^{-j\beta R_1} dz', \tag{6}$$

where

$$R_1 = \sqrt{(z - z')^2 + a^2}, \tag{7}$$

and $I_z' \equiv I_z(z')$ is the axial current at z'.

The integral equation for the current, originally derived by Hallén,[2] is[10]

$$\begin{aligned} \frac{4\pi}{\Pi} A_z &= \int_{-h}^{h} I_z' R_1^{-1} e^{-j\beta R_1} dz' \\ &= \frac{-j4\pi}{R_c}\left[C_1 \cos\beta z + \tfrac{1}{2} V_0^e \sin\beta\,|z| - z^i \int_0^z I(s)\sin\beta(z - s)\,ds\right]. \end{aligned} \tag{8}$$

V_0 is the driving potential difference maintained by the slice generator at $z = 0$; C_1 is a constant of integration which is later evaluated using (2); $R_c = c\Pi = 376.7$ ohms $\doteq 120\pi$ ohms. In practice the conductivity σ is usually sufficiently high and therefore z^i sufficiently small so that the last integral in (8) contributes negligibly to the final result.[11] For simplicity it is omitted throughout the following analysis. If required it can be included readily at appropriate points with no change in the formulation.

2. Expansion of the integral equation. In the absence of an exact solution of the integral equation (8) in closed form, an approximate solution may be obtained by expanding the integral on the left in a converging power series in terms of an appropriately chosen parameter. If a converging series is obtained and a sufficient number of terms can be evaluated the choice of the parameter for expansion is unimportant. If only a few terms in the series can be evaluated readily it is of great importance to select the parameter in such a way that convergence is so rapid that the sum of two or three terms gives a satisfactory approximation. The several parameters which have been used,[9,12] including that introduced below, will be discussed critically and results compared in another paper. The general definition of all such parameters is formulated below.

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[8] R. King, *Electromagnetic engineering* Vol. 1, McGraw-Hill Book Co., New York, 1945, p. 241.

[9] S. A. Schelkunoff, *Electromagnetic waves*, D. Van Nostrand Co., New York, 1943, pp. 140, 142 ff.

[10] Reference 3, equation (25). The complete derivation is given.

[11] R. King and F. G. Blake, Proc. I.R.E. **30**, 335 (1942).

[12] M. C. Gray, J. Appl. Phys. **15**, 61 (1944).

The solution of (8) may be formulated by expressing I_z in terms of a convenient reference current such as the input current I_0 and a distribution function $f(z)$ that is unknown. Thus let

$$I_z = I_0 f(z); \qquad I_z' = I_0 f(z'), \tag{9a}$$

so that

$$I_z' = I_z f(z')/f(z) \equiv I_z g(z, z'). \tag{9b}$$

The relative distribution function $g(z, z')$ is defined in (9b). Now let a function $\Psi(z)$ be defined by

$$\Psi(z) \equiv \int_{-h}^{h} g(z, z') R_1^{-1} e^{-i\beta R_1} dz'. \tag{10}$$

If the relative distribution function $g(z, z')$ were the actual one, it would be correct to write $I_z\Psi(z)$ for the integral on the left in (8). Whatever the form of $g(z, z')$, it is correct to write

$$\frac{4\pi}{\Pi} A_z = \int_{-h}^{h} I_z' R_1^{-1} e^{-i\beta R_1} dz' = I_z \Psi(z) + \int_{-h}^{h} [I_z' - I_z g(z, z')] R_1^{-1} e^{-i\beta R_1} dz'. \tag{11}$$

The more nearly $g(z, z')$ approximates the true distribution the smaller will be the difference integral on the right in (11). If $g(z, z')$ can be chosen accurately enough so that the integral on the right in (11) is considerably smaller than the term $I_z\Psi(z)$ for all values of z, it is possible to treat this term as the principal part and the difference integral as a correction.

If $g(z, z')$ were the true relative distribution function so that the difference integral in (11) were zero, the function $\Psi(z)$ would be given by

$$\Psi(z) = \frac{4\pi}{\Pi} \frac{A_z}{I_z}. \tag{12}$$

That is, $\Psi(z)$ would be proportional to the ratio of the vector potential on the surface of the antenna at a point z divided by the total axial current at z. It is clear from (6) that the vector potential at a point z is determined largely by the current at and near z, except possibly at a few points where I_z is very small compared with the currents elsewhere in the antenna. It may be assumed, therefore, that the ratio A_z/I_z is reasonably constant and predominantly real at all points along the antenna except at and near very small or zero values of the current. Clearly, since $I_z=0$ at the ends and A_z is not zero there, $\Psi(z)$ is infinite at $z=\pm h$. However, the product $I_z\Psi(z)$ must remain finite and relatively small at $z=\pm h$.

If $\Psi(z)$ is sensibly constant for most values of z, it must be exactly equal to $\Psi(z_0)$ at some point $z=z_0$, so chosen that $\Psi(z_0)$ is a good approximation of $\Psi(z)$ except where I_z is small or zero. Let

$$\Psi = |\Psi(z_0)|, \tag{13a}$$

so that

$$\Psi(z) = \Psi e^{i\theta_\Psi}. \tag{13b}$$

Also let a function $\gamma(z)$ be defined so that

$$\Psi(z) = \Psi + \gamma(z), \tag{14a}$$

where

$$\gamma(z) = \Psi(e^{j\theta_\Psi} - 1). \tag{14b}$$

If $\Psi(z)$ is predominantly real, $\gamma(z)$ is a small complex correction function except at values of z where I_z is small or zero. It is to be noted that $\gamma(z)$ is infinite at $z = \pm h$ but that $I_z\gamma(z)$ is finite and small there.

If $g(z, z')$ is not the true relative distribution function but only approximate, (12) is also approximate, and it is still possible to write (13a, b) and (14a, b) with $\Psi(z)$ defined as in (10). Substituting (14a) in (11) and using (11) in (8) solved for I_z in the principal term $I_z\Psi$, one obtains

$$I_z = \frac{-j4\pi}{R_c\Psi}\{C_1 \cos \beta z + \tfrac{1}{2}V_0^e \sin \beta |z|\} - \frac{1}{\Psi}\left\{I_z\gamma(z) + \int_{-h}^{h} [I_z' - I_z g(z, z')]R_1^{-1}e^{-j\beta R_1}dz'\right\}. \tag{15}$$

This equation is exact. Like (8) it is an integral equation in the current, but the current appears in the integrand of a difference integral that is small. The term $I_z\gamma(z)$ is also small except near points where I_z is small or vanishes, as at $z = \pm h$.

A more useful form of (15) is obtained as follows. Let (8) be written with $z = h$ in the form

$$0 = \frac{-j4\pi}{R_c\Psi}\{C_1 \cos \beta h + \tfrac{1}{2}V_0^e \sin \beta h\} - \frac{1}{\Psi}\int_{-h}^{h} I_z' R_{1h}e^{-j\beta R_{1h}}dz'. \tag{16}$$

The term in z^i has been omitted in (16) just as in (15). Actually (16) is exactly equivalent to (15) when this is written with $z = h$. In (16)

$$R_{1h} = \sqrt{(h - z')^2 + a^2}. \tag{17}$$

The desired equation is obtained by subtracting (16) from (15). It is

$$I_z = \frac{-j4\pi}{R_c\Psi}\{C_1[\cos \beta z - \cos \beta h] + \tfrac{1}{2}V_0^e[\sin \beta |z| - \sin \beta h]\} - \frac{1}{\Psi}\left\{I_z\gamma(z) + \int_{-h}^{h} [I_z' - I_z g(z, z')]R_1^{-1}e^{-j\beta R_1}dz' - \int_{-h}^{h} I_z' R_{1h}^{-1}e^{-j\beta R_{1h}}dz'\right\}. \tag{18}$$

This is the final exact form of the integral equation. Its principal advantage over (8) lies in the fact that all terms on the right involving the current are small if the relative distribution function $g(z, z')$ is correctly chosen to make the difference terms small. The expression (18) must be used in preference to (15) because in (18) the right side vanishes *for all values* of βh when $z = \pm h$ as required by (2), whereas the right side in (15) can not be made to vanish at $x = \pm h$ when $\cos \beta h = 0$. In this case the arbitrary constant C_1 disappears from (15).

The integral equation (18) can be expressed as the sum of a principal current $(I_z)_0$ consisting of the trigonometric terms and a correction current $(I_z)_c$ given by the remaining terms. The correction term $(I_z)_c$ can then be expanded in a power series in $1/\Psi$. Thus

$$I_z = (I_z)_0 + (I_z)_c = (I_z)_0 + (I_z)_{c_1} + (I_z)_{c_2} + (I_z)_{c_3} + \cdots . \tag{19}$$

Here $(I_z)_{c_1}$ is obtained by substituting $(I_z)_0$ in $(I_z)_c$; $(I_z)_{c_1}+(I_z)_{c_2}$ is obtained using $(I_z)_0+(I_z)_{c_1}$ in $(I_z)_c$, etc.

For convenience let

$$F_n(z) - F_n(h) \equiv F_{nz}; \qquad G_n(z) - G_n(h) \equiv G_{nz} \tag{20a}$$

where

$$F_0(z) \equiv \cos\beta z; \quad F_0(h) \equiv \cos\beta h; \quad G_0(z) \equiv \sin\beta|z|; \quad G_0(h) \equiv \sin\beta h \tag{20b}$$

$$F_n(z) \equiv F_{n-1,z}\int_{-h}^{h} g(z, z')R_1^{-1}e^{-j\beta R_1}dz' - \int_{-h}^{h} F_{n-1,z'}R_1^{-1}e^{-j\beta R_1}dz' - F_{n-1,z}\gamma(z), \tag{21a}$$

$$F_n(h) \equiv -\int_{-h}^{h} F_{n-1,z'}R_{1h}^{-1}e^{-j\beta R_{1h}}dz'. \tag{21b}$$

The first and last terms in (21a) may be combined into $F_{n-1,z}\Psi$ using (10) and (14a). Expressions for $G_n(z)$ and $G_n(h)$ are obtained from (21a) and (21b) by writing G for F throughout.

Using (19)–(21) in (18), the complete series solution for I_z may be obtained. The constant C_1 may be evaluated from (16) using (19)–(21) as described in references 2 and 3. The resulting mth order current is[13]

$$(I_z)_m = \frac{j2\pi V_0^e}{R_c\Psi}\left\{\frac{\sum_{n=0}^{m} F_n(z)/\Psi^n\cdot\sum_{n=0}^{m} G_n(h)/\Psi^n - \sum_{n=0}^{m} G_n(z)/\Psi^n\cdot\sum_{n=0}^{m} F_n(h)/\Psi^n}{\sum_{n=0}^{m} F_n(h)/\Psi^n}\right\}. \tag{22}$$

This formula may be simplified using (20a, b). The result is

$$(I_z)_m = \frac{j2\pi V_0^e}{R_c\Psi}\left\{\frac{\sin\beta(h-|z|) + \sum_{n=1}^{m} M_n(z)/\Psi^n}{\cos\beta h + \sum_{n=1}^{m} F_n(h)/\Psi^n}\right\}, \tag{23}$$

where, in particular,

$$\begin{aligned} M_1(z) \equiv M_1^{I}(z) + jM_1^{II}(z) &= F_1(z)\sin\beta h - F_1(h)\sin\beta|z| + G_1(h)\cos\beta z \\ &\quad - G_1(z)\cos\beta h, \end{aligned} \tag{24}$$

$$\begin{aligned} M_2(z) \equiv M_2^{I}(z) + jM_2^{II}(z) &= F_2(z)\sin\beta h - F_2(h)\sin\beta|z \\ &\quad + G_1(h)F_1(z) - G_1(z)F_1(h) + G_2(h)\cos\beta z - G_2(z)\cos\beta h, \end{aligned} \tag{25}$$

and, as previously defined,[2,4,5,6]

$$F_n(h) = \alpha_n \equiv \alpha_n^{I} + j\alpha_n^{II}. \tag{26}$$

With

[13] C. J. Bouwkamp, Physica 9, 609 (1942). In Bouwkamp's paper G and F are, respectively, the F and G functions in this analysis.

$$\beta_n = \beta_n^I + j\beta_n^{II} \equiv M_n(0) = M_n^I(0) + jM_n^{II}(0), \tag{27}$$

the impedance of the antenna is defined by

$$Z_0 = V_0^e/I_0, \tag{28}$$

where I_0 is given by (23) with $z=0$. It is

$$(Z_0)_m = (R_0)_m + j(X_0)_m = \frac{-jR_c\Psi}{2\pi}\left\{\frac{\cos\beta h + \sum_{n=1}^{m}\alpha_n/\Psi^n}{\sin\beta h + \sum_{n=1}^{m}\beta_n/\Psi^n}\right\}. \tag{29}$$

This is a generalization of the formula obtained by Hallén[2] and others[3,13], as shown later.

3. Functions and parameters in the Hallén solution. The expressions for the current (23) and for the impedance (29) depend upon the constant parameter Ψ, and this in turn depends upon the relative distribution function $g(z, z')$. The definition of these quantities involves the following considerations: The relative distribution function $g(z, z')$ must be so chosen that it is a sufficiently good approximation of the actual distribution to make the difference integral in (11) small. Furthermore, it must be sufficiently simple in form that the integral (10) for $\Psi(z)$ can be evaluated and separated into a principal, constant, real part $|\Psi(z_0)| \equiv \Psi$ and a small correction term $\gamma(z)$ as in (14a, b).

The choice of distribution function made by Hallén depended upon the reasonable albeit implicit assumption that the vector potential A_z at z depends primarily upon the current at and near z. If contributions from all more distant elements of current are small, A_z may be evaluated approximately by assuming the current at all points to be I_z and neglecting retardation. This is equivalent to setting

$$g_H(z, z') = e^{j\beta R_1}. \tag{30}$$

The subscript H will be used to designate parameters and functions in the Hallén analysis. With (30), (10) gives

$$\Psi_H(z) = \int_{-h}^{h}\frac{dz'}{R_1} = \sinh^{-1}\frac{h+z}{a} + \sinh^{-1}\frac{h-z}{a}\cdot \tag{31}$$

Alternatively and equivalently

$$\Psi_H(z) = \Omega + \ln\left(1 - \frac{z^2}{h^2}\right) + \delta(z), \tag{32}$$

where

$$\Omega = \Psi_H(z=0) = 2\ln\frac{2h}{a}, \tag{33}$$

$$\delta(z) = \ln\left\{\frac{1}{4}\left[\sqrt{1+\left(\frac{a}{h+z}\right)^2}+1\right]\left[\sqrt{1+\left(\frac{a}{h-z}\right)^2}+1\right]\right\}. \tag{34}$$

If the function $\Psi_H(z)$ in (31) is plotted as a function of z/a for a range of values of the ratio h/a, it is found to be moderately constant for large ratios h/a except with

z near h. Subject to the condition $a^2 \ll h^2$, the maximum value of $\Psi_H(z)$ is Ω, the average value is $\Omega-2+2\ln 2=\Omega-0.614$, the value at $z=\pm h$ is $\frac{1}{2}\Omega+\ln 2$. For $\Omega>15$ the average value or the maximum value are satisfactory approximations. In view of the fact that $\Psi_H(z)$ becomes smaller at $z=\pm h$ instead of becoming infinite as it would if the correct distribution function were used, it is clear that the curvature of $\Psi_H(z)$ is the reverse of what it should be. Therefore, the maximum value Ω is probably the best approximation of $\Psi_H(z)$ and this was Hallén's, although not explicitly for this reason. Thus the Hallén analysis sets

$$\Psi_H(z_0) \equiv \Psi_H = \Omega = 2 \ln \frac{2h}{a}, \tag{35a}$$

$$\gamma_H(z) = \ln\left(1 - \frac{z^2}{h^2}\right) + \delta(z). \tag{35b}$$

The Hallén expressions for the current and the impedance are given by (23) and (29) with Ω written for Ψ and with appropriately modified functions $F_n(z)$, $F_n(h)$, $G_n(z)$, and $G_n(h)$, $n>0$. The functions with $n=0$ are independent of the choice of Ψ. The Hallén functions are

$$F_{nH}(z) = (F_{n-1,z})_H\Omega - \int_{-h}^{h} (F_{n-1,z'})_H \frac{e^{-j\beta R_1}}{R_1} dz', \tag{36a}$$

$$F_{nH}(h) = - \int_{-h}^{h} (F_{n-1,z'})_H \frac{e^{-j\beta R_{1h}}}{R_{1h}} dz'. \tag{36b}$$

$G_{nH}(z)$ and $G_{nH}(h)$ are obtained from the above by writing G for F. These functions have been evaluated elsewhere[3,13] for $n=1$ and $n=2$. The first order distribution of current and the first order impedance have been calculated and represented graphically[3,11]; the second order impedance has been evaluated by Bouwkamp.[13] The Hallén formula for the mth order current is

$$(I_z)_{mH} = \frac{j2\pi V_0^e}{R_c\Omega}\left\{\frac{\sin\beta(h-|z|) + \sum_{n=1}^{m} M_{nH}(z)/\Omega^n}{\cos\beta h + \sum_{n=1}^{m} F_{nH}(h)/\Omega^n}\right\}, \tag{37}$$

$$(Z_0)_{mH} = \frac{-jR_c\Omega}{2\pi}\left\{\frac{\cos\beta h + \sum_{n=1}^{m} \alpha_{nH}/\Omega^n}{\sin\beta h + \sum_{n=1}^{m} \beta_{nH}/\Omega^n}\right\}. \tag{38a}$$

Here

$$\alpha_{nH} = \overset{I}{\alpha_n} + j\overset{II}{\alpha_n} = F_{nH}(h), \quad (38b) \qquad \beta_{nH} = \overset{I}{\beta_n} + j\overset{II}{\beta_n} = M_{nH}(0). \quad (38c)$$

The functions α_1 and β_1 are tabulated and represented graphically in references 2, 3, 11; the functions α_2 and β_2 as calculated by Bouwkamp[13] using graphical methods are listed in Table I and plotted in Figs. 2 and 3.

4. Functions and parameters in the improved solution. The relative distribution function $g(z, z')$ in (30) is the simplest and the most obvious one if an attempt is

TABLE I

βh	α_2^I	α_2^{II}	β_2^I	β_2^{II}
0	0	0	0	0
0.2	−0.16	0.03	3.07	—
0.4	−0.53	0.13	5.20	0.03
0.6	−1.07	0.39	6.50	0.24
0.8	−1.67	0.80	7.14	0.78
1.0	−2.17	1.31	6.78	1.74
1.2	−2.66	1.84	5.48	3.04
1.4	−3.00	2.31	3.34	4.97
1.6	−3.23	2.73	0.45	7.06
1.8	−3.34	3.04	−3.06	9.33
2.0	−3.33	3.30	−7.03	11.81
2.2	−3.16	3.48	−11.25	13.98
2.4	−2.84	3.58	−16.22	16.08
2.6	−2.35	3.58	−20.83	17.28
2.8	−1.59	3.40	−24.71	17.72
3.0	−0.61	2.99	−27.54	17.50
3.2	0.50	2.27	−29.02	16.85
3.4	1.58	1.37	−29.29	15.75
3.6	2.59	0.28	−28.30	13.84
3.8	3.49	−0.83	−26.35	11.34
4.0	4.33	−2.00	−23.38	8.28
4.2	5.03	−3.09	−19.60	4.73
4.4	5.41	−4.13	−15.14	0.21
4.6	5.46	−5.04	−10.26	−4.84
4.8	5.20	−5.67	−4.21	−10.09
5.0	4.66	−6.08	+2.41	−15.10

made to solve the original integral equation (8) as was done by Hallén. On the other hand, if the formal solution is carried through to obtain (23) without previously selecting $g(z, z')$ as has been done in the present analysis, it is perfectly clear that the leading term in the distribution of current for any value of the distribution function must be of the form

$$I_z = Kf_1(z) \tag{39a}$$

with

$$f_1(z) \equiv \sin \beta(h - |z|). \tag{39b}$$

K is an amplitude factor independent of z. Accordingly, an approximate relative distribution function is

$$g_{K1}(z, z') = \frac{\sin \beta(h - |z'|)}{\sin \beta(h - |z|)} = \frac{f_1(z')}{f_1(z)}. \tag{40}$$

This function is known to be a very much better approximation of the actual current then the function assumed by Hallén, $g_H(z, z') = e^{i\beta R_1}$. The function $f_1(z) = \sin \beta(h - |z|)$ *actually is proportional to the principal part of the current*; *the function* $e^{i\beta R_1}$ *is not.* Using (40) and (10) we obtain

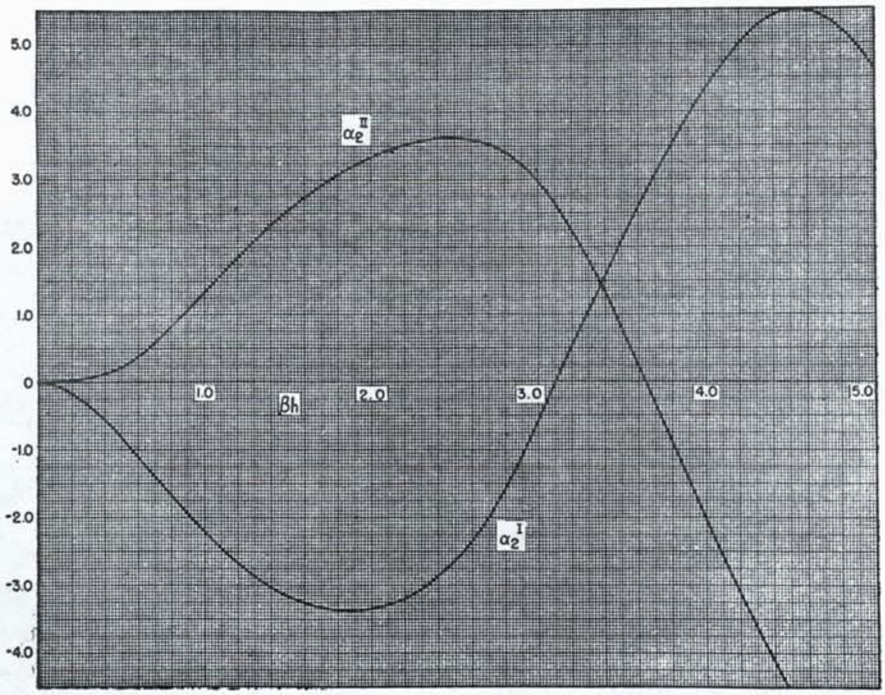

FIG. 2. The parameters α_2^I and α_2^{II} as a function of βh.

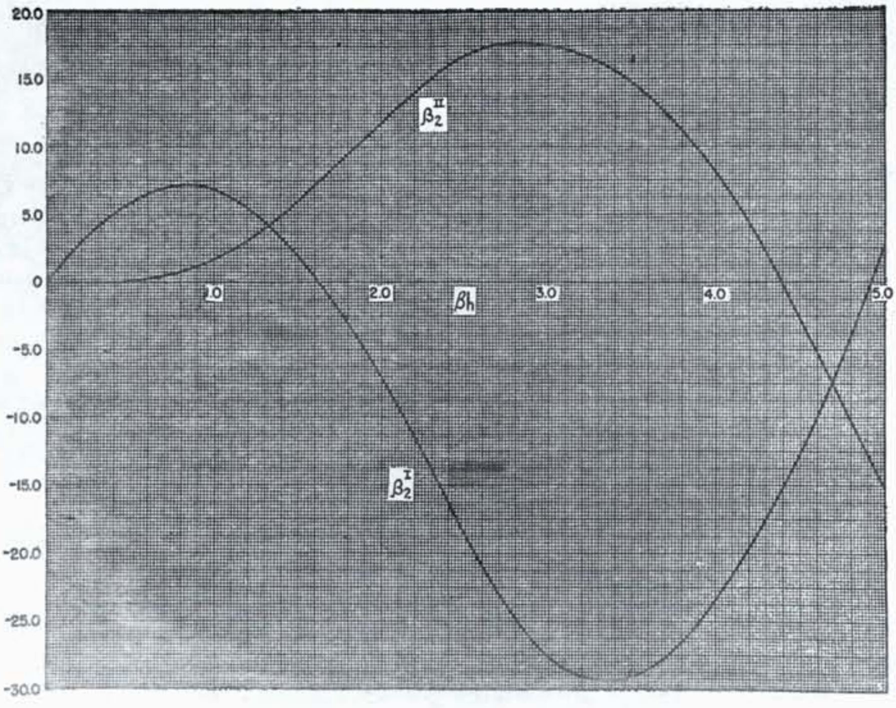

FIG. 3. The parameters β_2^I and β_2^{II} as a function of βh.

$$\Psi_{K1}(z) = \int_{-h}^{h} g_{K1}(z, z')R_1^{-1}e^{-i\beta R_1}dz'. \tag{41}$$

This function involves the factor $f_1(z) = \sin \beta(h - |z|)$ in the denominator of the integrand. Since this is not a function of z' it is a constant in the integration. Therefore, it is convenient to introduce the function

$$\psi_1(z) \equiv \int_{-h}^{h} f_1(z')R_1^{-1}e^{-i\beta R_1}dz' \tag{42a}$$

so that

$$\Psi_{K1}(z) = \frac{\psi_1(z)}{f_1(z)}. \tag{42b}$$

The function $\psi_1(z)$ can be written in the form

$$\Psi_1(z) = C(z) \sin \beta h - S(z) \cos \beta h, \tag{43}$$

where

$$C(z) \equiv \int_{-h}^{h} \cos \beta z' R_1^{-1}e^{-i\beta R_1}dz', \tag{44}$$

$$S(z) \equiv \int_{-h}^{h} \sin \beta |z'| R_1^{-1}e^{-i\beta R_1}dz'. \tag{45}$$

These integrals are evaluated in the Appendix both in general and in a simpler approximate form. The latter is a good approximation if, as assumed throughout this analysis, $h^2 \gg a^2$. Curves for $C(z)$ and $S(z)$ as calculated using the simpler forms which apply in this analysis are given in Figs. 4–7, 20–23 for $\beta h = \pi/2$ and π and for $\Omega = 2 \ln (2h/a) = 10$ and 20. It is to be noted that

$$\psi_1(z) = C(z); \qquad \beta h = \pi/2, \tag{46}$$

$$\psi_1(z) = S(z); \qquad \beta h = \pi. \tag{47}$$

It follows that the plots of $C(z)$ with $\beta h = \pi/2$ are also plots of $\psi_1(z)$; these are given in Figs. 4 and 5 for $\Omega = 10$ and 20. Similarly plots of $S(z)$ with $\beta h = \pi$ are also plots of $\psi_1(z)$; these are given in Figs. 6 and 7 for $\Omega = 10$ and 20. The function $\psi_1(z)$ is seen to have a very small imaginary part so that it and $\Psi_{K1}(z) = \psi_1(z)/f_1(z)$ are predominantly real, in confirmation of the assumption made in conjunction with (14). Accordingly, the parameter $\Psi = |\Psi(z_0)|$ defined in (13a) may be chosen to be

$$\Psi_{K1} = |\Psi_{K1}(0)| = |\psi_1(0)|; \qquad \beta h = \pi/2; \tag{48}$$

$$\Psi_{K1} = |\Psi_{K1}(h - \lambda/4)| = |\psi_1(h - \lambda/4)|; \qquad \beta h = \pi. \tag{49}$$

The function

$$|\Psi_{K1}(z)| = \frac{|\psi_1(z)|}{f_1(z)} \tag{50}$$

is plotted in Figs. 4–7. For $\beta h = \pi/2$ and both for $\Omega = 10$ and 20 it is seen to be quite constant over the entire length of the antenna except near the ends where it becomes infinite, as it should. For $\beta h = \pi$ the function becomes infinite not only at the ends but also at the center. The infinity at the center is a result of approximating the

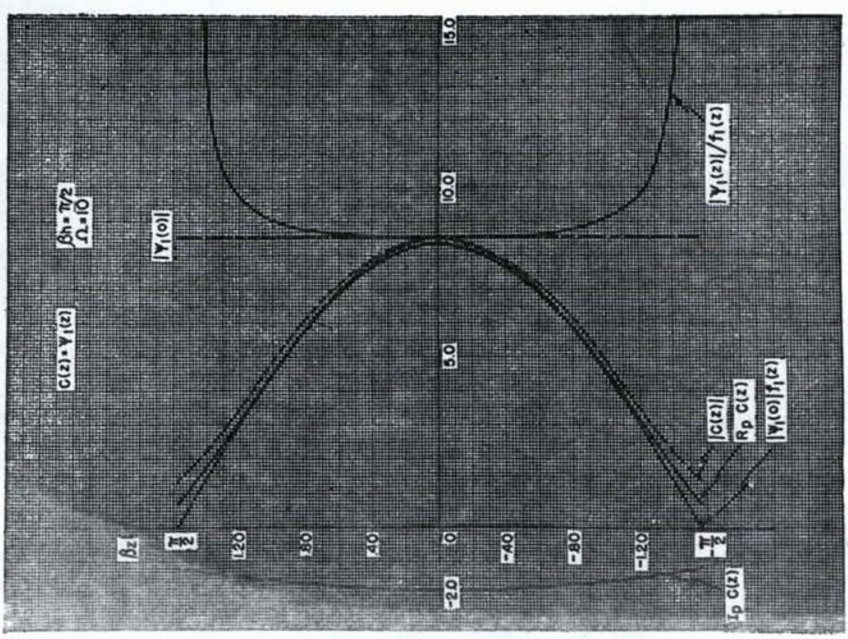

FIG. 4. The functions $C(z)=\psi_1(z)$; $|\psi_1(0)|f_1(z)$, and $|\psi_1(z)|/f_1(z)$ near resonance, $\beta h=\pi/2$, $\Omega=10$.

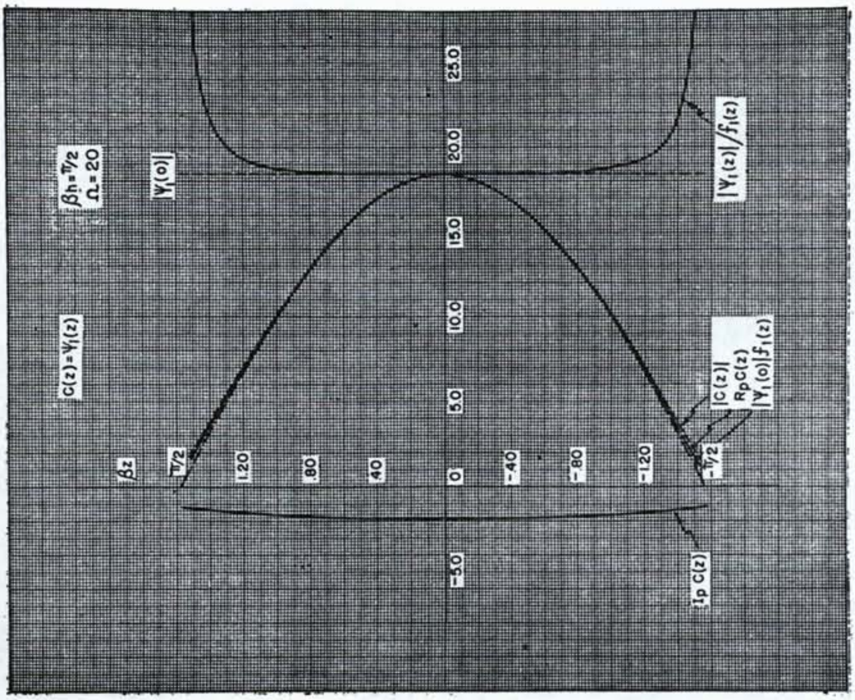

FIG. 5. The functions $C(z)=\psi_1(z)$, $|\psi_1(0)|f_1(z)$, and $|\psi_1(z)|/f_1(z)$ near resonance, $\beta h=\pi/2$, $\Omega=20$.

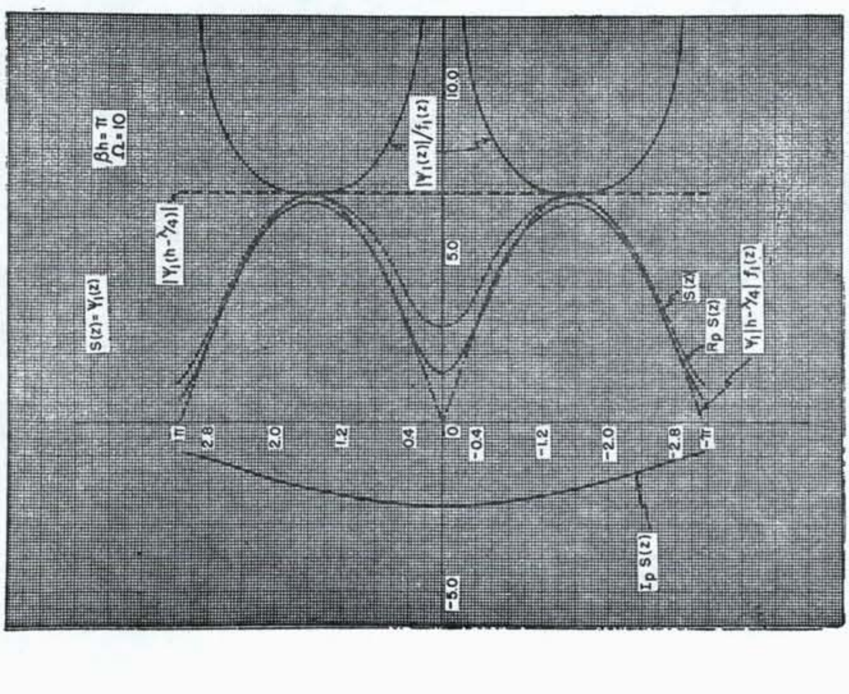

FIG. 6. The functions $S(z)=\psi_1(z)$, $|\psi_1 h-\lambda/4)|f_1(z)$, and $|\psi_1(z)|/f_1(z)$ near anti-resonance, $\beta h=\pi$, $\Omega=10$.

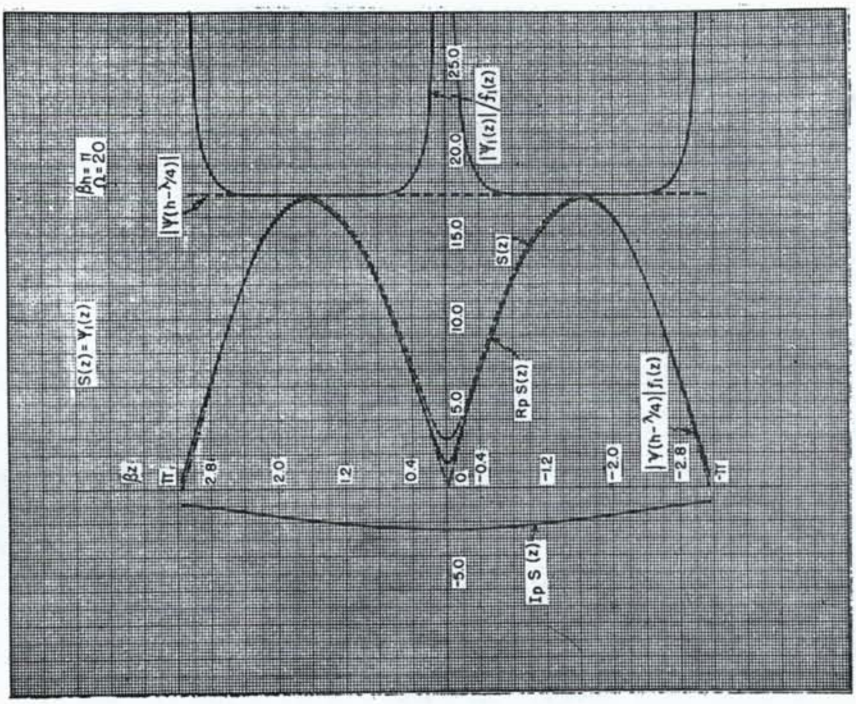

FIG. 7. The function $S(z)=\psi_1(z)$, $|\psi_1(h-\lambda/4)|f_1(z)$, and $|\psi_1(z)/f_1(z)$ near anti-resonance, $\beta h=\pi$, $\Omega=20$.

current by the distribution function $f_1(z)=\sin\beta(h-|z|)$. With $\beta h=\pi$ and $z=0$, this vanishes so that $\Psi_{K1}(z)$ necessarily diverges. Unlike the infinity at the ends, the infinity at $z=0$ is due to the fact that $f_1(z)$ and hence $g_K(z, z')$ are approximate and not exact distribution functions. Actually, the *current does not vanish* at $z=0$; it merely is small so that $\Psi(z)$ does not become infinite. The fact that I_z is small at and near $z=0$ does not mean that $\Psi(z)$ necessarily becomes very large. $\Psi(z)$ is by definition proportional to the ratio A_z/I_z, and A_z is determined largely by the current at z. Hence $A_{z=0}$ is small if $I_{z=0}$ is, and the ratio may remain moderately constant. Furtheremore, since A_z at $z=h-\lambda/4$ is determined principally by the large (near maximum) currents at and near $z=h-\lambda/4$, it is affected only very slightly by a small current at $z=0$. Therefore A_z at $z=h-\lambda/4$ and $\Psi_{K1}(h-\lambda/4)$ will not be sensibly different if a fictitious zero current is assumed at $z=0$ or an actual small current. Accordingly the function $|\Psi_{K1}(h-\lambda/4|$ is a good approximation of $\Psi_{K1}(z)$ for the *actual* current everywhere (including $z=0$) except near the ends, $z=\pm h$.

Although the qualitative argument to show that $\Psi_{K1}(z)$ is sensibly constant and approximately equal to $|\Psi_{K1}(h-\lambda/4)|$ for all values of z except the ends is sound, it can be verified directly using Hallén's first order distribution. It has been shown[3] that a very satisfactory approximation of the Hallén first order current is given by

$$I_z = I_0'' \left(\frac{\cos\beta z - \cos\beta h}{1-\cos\beta h}\right) + jI_m' \sin\beta(h-|z|); \quad \frac{\pi}{2} \leqq \beta h < 2\pi, \tag{51}$$

where I_0'' is the component of current at $z=0$ in phase with the driving potential difference and I_m' is the maximum value of the component of current in phase quadrature with the driving potential difference. I_m' occurs at $z=h-\lambda/4$. With

$$k = I_0''/I_m'; \qquad |k| < 1, \tag{52}$$

it is possible to write (51) in the form

$$I_z = jI_m' \left\{\sin\beta(h-|z|) - jk\left(\frac{\cos\beta z - \cos\beta h}{1-\cos\beta h}\right)\right\} \equiv jI_m' f_2(z). \tag{53}$$

With this approximate current, an appropriate distribution function $g(z, z')$ is defined by

$$\begin{aligned} g_2(z, z') = \frac{I_z'}{I_z} &= \frac{\sin\beta(h-|z'|) - jk(\cos\beta z' - \cos\beta h)/(1-\cos\beta h)}{\sin\beta(h-|z|) - jk(\cos\beta z - \cos\beta h)/(1-\cos\beta h)} \\ &\equiv \frac{f_2(z')}{f_2(z)}; \qquad \frac{\pi}{2} \leqq \beta h < 2\pi. \end{aligned} \tag{54}$$

The ratio factor k is negative and small compared with unity. It is plotted in Fig. 8 as a function of βh from the data of Figs. 9–11 in reference 3. Only values of βh near π are used because for βh not near integral multiples of π the distribution $(\cos\beta z - \cos\beta h)$ does not differ greatly from $\sin\beta(h-|z|)$. At $\beta h=\pi/2$ they are identical.

Using the notation (42a, b),

$$\psi_2(z) \equiv \int_{-h}^{h} f_2(z') R_1^{-1} e^{-j\beta R_1} dz', \tag{55a}$$

so that

$$\Psi_{K2}(z) = \frac{\psi_2(z)}{f_2(z)}. \tag{55b}$$

The function $\psi_2(z)$ can be written

$$\psi_2(z) = C(z) \sin \beta h - S(z) \cos \beta h - jk \left[\frac{C(z) - E(z) \cos \beta h}{1 - \cos \beta h} \right], \tag{56}$$

where $C(z)$ and $S(z)$ are defined in (44) and (45), and $E(z)$ is given by

$$E(z) \equiv \int_{-h}^{h} R_1^{-1} e^{-j\beta R_1} dz'. \tag{57}$$

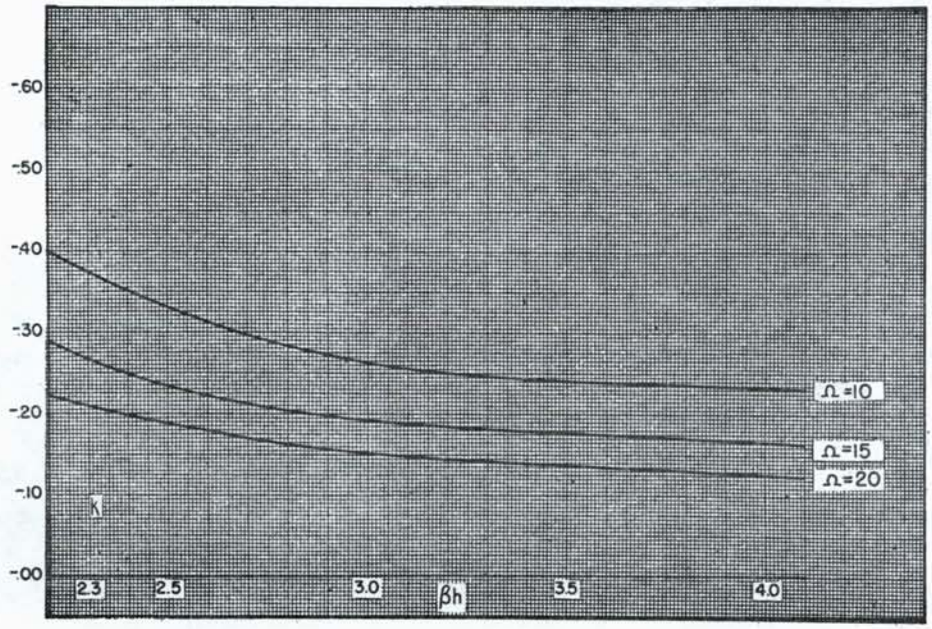

FIG. 8. The quantity $k = I_0''/I_m'$ as a function of βh near anti-resonance.

This function is evaluated in the Appendix both in general and in a simpler approximate form valid when it is possible to write $a^2 \ll h^2$, as in the present analysis. $E(z)$ is plotted in Figs. 24 and 25 for $\beta h = \pi/2$ and π and with $\Omega = 10$ and 20. The function $\psi_2(z)$ is necessarily predominantly real because it is known that the first two terms in (56)—these are identically $\psi_1(z)$—are predominantly real and that k is small. The functions $|\psi_2(z)|$ and $|\Psi_{K2}(z)| = |\psi_2(z)|/|f_2(z)|$ are shown in Figs. 9 and 10 for $\beta h = \pi$ and $\Omega = 10$ and 20. It is seen that $|\Psi_{K2}(z)|$ does not become infinite at $z = 0$, and is reasonably constant and equal to $|\Psi_{K2}(h - \lambda/4)| \doteq |\psi_2(h - \lambda/4)|$ for all values of z except at the ends where it becomes infinite, as it should. Comparison of Figs. 9 and 10 with 6 and 7 shown that $|\psi_2(h - \lambda/4)|$ differs only slightly from $|\psi_1(h - \lambda/4)|$. The difference is greater for the smaller value of Ω. It follows that $|\psi_1(h - \lambda/4)|$ is a satisfactory parameter even for $\beta h = \pi$. If desired $|\psi_2(h - \lambda/4)|$ may be used especially for small values of Ω, but the difference is not over about 3% for $\Omega \geqq 10$.

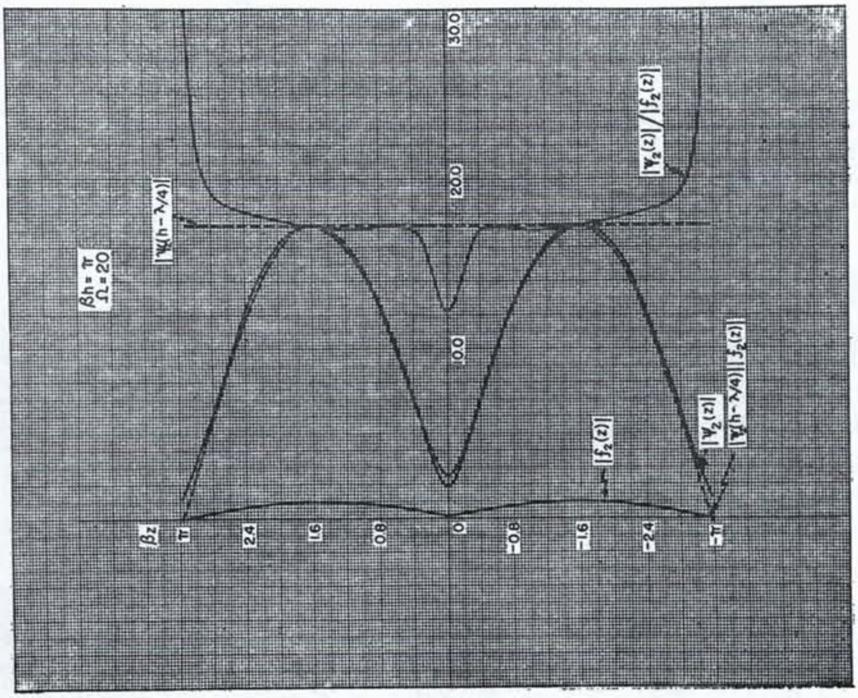

FIG. 10. The functions $|f_2(z)|$, $|\psi_2(h-\lambda/4)||f_2(z)|$, and $|\psi_2(z)|/|f_2(z)|$ near anti-resonance, $\beta h=\pi$, $\Omega=20$.

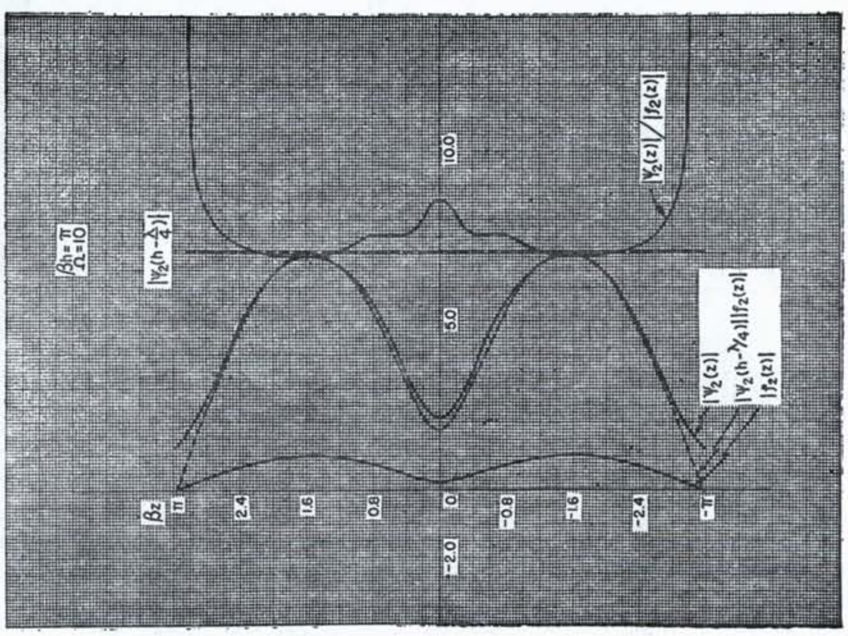

FIG. 9. The functions $|f_2(z)|$, $|\psi_2(h-\lambda/4)||f_2(z)|$, and $|\psi_2(z)|/|f_2(z)|$ near anti-resonance, $\beta h=\pi$, $\Omega=10$.

The conclusions of the above analysis may be generalized and summarized as follows: 1. The relative distribution function

$$g_{1K}(z, z') = \frac{\sin \beta(h - |z'|)}{\sin \beta(h - |z|)}$$

is a good approximation for all values of h. 2. Suitable parameters for expansion are

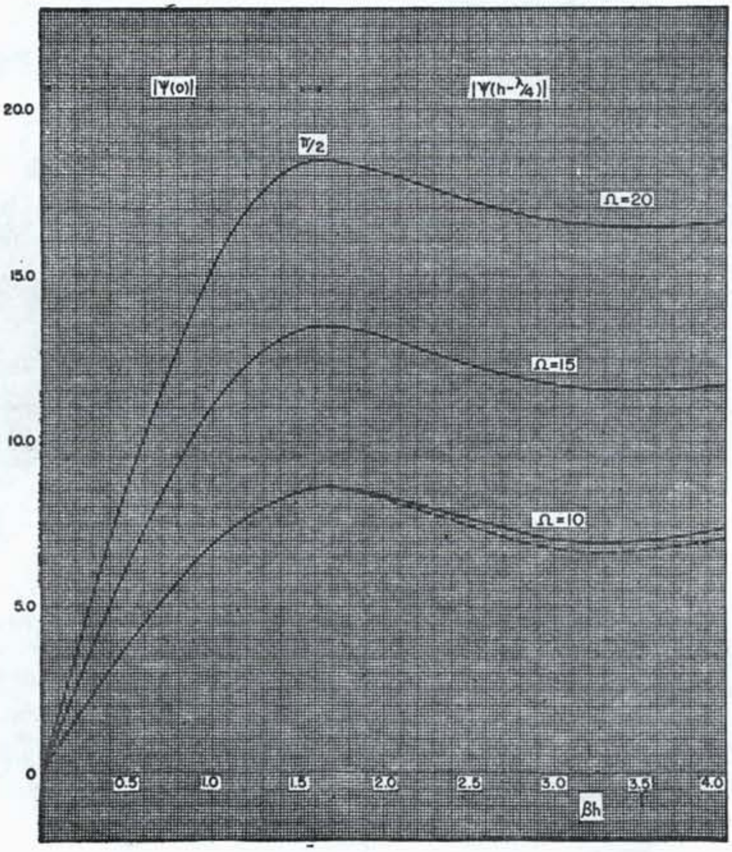

FIG. 11. The parameter ψ, Eqs. (48) and (49), as a function of βh, for $\Omega=10, 15, 20$.

$|\Psi_{K1}(0)| = |\psi_1(0)|$ for $\beta h \leqq \pi/2$; $|\Psi_{K1}(h-\lambda/4)| = |\psi_1(h-\lambda/4)|$ for $\beta h \geqq \pi/2$. The following notation will be used from here on

$$\psi \equiv \Psi = \begin{cases} |\Psi_{K1}(0)| = |\psi_1(0)|; & \beta h \leqq \pi/2 \\ |\Psi_{K1}(h - \lambda/4)| = |\psi_1(h - \lambda/4)|; & \beta h \geqq \pi/2. \end{cases} \tag{58}$$

Since $\Psi_{K1}(z)$ has such a small imaginary part and is so well represented by (58) except at the ends of the antenna, the correction function $\gamma(z)$ in (14a) is sufficiently small to be neglected.

The parameter ψ as defined in (58) is plotted as a function of βh for $\Omega=10, 15, 20$

in Fig. 11. For $\Omega=10$ the curve in solid line for $\beta h>\pi/2$ is $|\psi_2(h-\lambda/4)|$, the curve in broken line is $|\psi_1(h-\lambda/4)|$. The curves for $\Omega=15$, and 20 with $\beta h>\pi/2$ are $|\psi_1(h-\lambda/4)|$.

5. Distribution of current. The distribution of current is given by (23), the impedance by (29) with $\Psi(=\psi)$ obtained from (58) or Fig. 11 for the value of h/a in question. The functions $F_0(z)$, $F_0(h)$, $G_0(z)$, and $G_0(h)$ are unchanged; they are defined by (20b). Upon substituting (40) in (10) and using (10) in the form (14a) in (21a) with $n=1$, this becomes

$$F_{1K}(z) = F_{0z}\Psi_{K1}(z) - \int_{-h}^{h} F_{0z'}R_1^{-1}e^{-i\beta R_1}dz'. \tag{59}$$

Upon comparing (59) with (36a) written for $n=1$, it follows that

$$F_{1K}(z) = F_{1H}(z) + (\psi - \Omega)(F_{0z})_H. \tag{60}$$

Since $\Psi(z)$ is not involved in (21b),

$$F_{1K}(h) = F_{1H}(h). \tag{61}$$

Upon substituting (60) in (21a) with $\gamma(z)=0$ and $n=2$, this becomes

$$F_{2K}(z) = (F_{1z})_K\Psi_{K1}(z) - \int_{-h}^{h} (F_{1z'})_K R_1^{-1}e^{-i\beta R_1}dz'. \tag{62}$$

Using (60) and (61) in (62), this gives

$$F_{2K}(z) = (F_{1z})_K\psi - \int_{-h}^{h} (F_{1z'})_H R_1^{-1}e^{-i\beta R_1}dz' - (\psi - \Omega)\int_{-h}^{h} F_{0z'}R_1^{-1}e^{-i\beta R_1}dz'. \tag{63}$$

Upon comparing (63) with (36a), this time written with $n=2$, and using (60) and (61) as well as (36a) written with $n=1$, we see that

$$\begin{aligned} F_{2K}(z) = {} & (F_{1z})_H\psi + \psi(\psi - \Omega)(F_{0z})_H + F_{2H}(z) - (F_{1z})_H\Omega \\ & - (\psi - \Omega)\left[(F_{0z})_H\psi - F_{1H}(z) - (\psi - \Omega)(F_{0z})_H\right]. \end{aligned}$$

When terms are collected there results,

$$F_{2K}(z) = F_{2H}(z) + (\psi - \Omega)(F_{1z})_H + (\psi - \Omega)F_{1H}(z) + (\psi - \Omega)^2(F_{0z})_H. \tag{64}$$

Using (21b) with (60) and (61), we find

$$F_{2K}(h) = F_{2H}(h) + (\psi - \Omega)F_{1H}(h). \tag{65}$$

Subtracting (65) from (64), we have

$$(F_{2z})_K = (F_{2z})_H + 2(\psi - \Omega)(F_{1z})_H + (\psi - \Omega)^2(F_{0z})_H. \tag{66}$$

Repetition of the above procedure to evaluate $(F_{nz})_K$ leads to:

$$(F_{nz})_K = \sum_{i=0}^{n} \frac{n!}{(n-i)!i!}(\psi - \Omega)^i(F_{n-i,z})_H; \qquad n \geqq 0. \tag{67}$$

Expressions for $G_{2K}(z)$, $G_{2K}(h)$, $(G_{2z})_K$, and $(G_{nz})_K$ are obtained from (64)–(67) by writing G for F.

If the functions F and G are combined to form the functions M as defined in (24) and (25) for the first two terms, the result is

$$M_{1K}(z) = M_{1H}(z) + (\psi - \Omega) \sin \beta(h - |z|), \tag{68}$$

$$M_{2K}(z) = M_{2H}(z) + 2(\psi - \Omega)M_{1H}(z) + (\psi - \mathrm{b})^2 \sin \beta(h - |z|). \tag{69}$$

In general,

$$M_{nK}(z) = \sum_{i=0}^{n} \frac{n!}{(n-i)!i!} (\psi - \mathrm{b})^i M_{n-i,H}(z); \qquad n \geqq 0, \tag{70}$$

where it is understood that

$$M_{0H}(z) = \sin \beta(h - |z|). \tag{71}$$

Similarly,

$$F_{nK}(h) = \sum_{i=0}^{n-1} \frac{(n-1)!}{(n-i-1)!i!} (\psi - \Omega)^i F_{n-i,H}(h); \qquad n \geqq 1. \tag{72}$$

Upon substituting (70)–(72) in (23) the general expression for the mth order current becomes

$$(I_z)_m = \frac{j2\pi V_0^e}{R_c\psi} \left\{ \frac{(D_1)_m \sin \beta(h - |z|) + \sum_{n=1}^{m} (D_{n+1})_m M_{nH}(z)/\psi^n}{\cos \beta h + \sum_{n=1}^{m} (D_n)_{m-1} F_{nH}(h)/\psi^n} \right\} \tag{73}$$

where with

$$x \equiv 1 - \frac{\Omega}{\psi} \tag{74}$$

the D's have the following significance:

Order m =	0	1	2	3	4		
$(D_1)_m$ =	1	$+ x$	$+ x^2$	$+ x^3$	$+ x^4$	$+ \cdots$	
$(D_2)_m$ =		1	$+ 2x$	$+ 3x^2$	$+ 4x^3$	$+ \cdots$	
$(D_3)_m$ =			1	$+ 3x$	$+ 6x^2$	$+ \cdots$	(75)
$(D_4)_m$ =				1	$+ 4x$	$+ \cdots$	
$(D_5)_m$ =					1	$+ \cdots$	

It is interesting and significant to note than when the series in (75) are summed for an infinite number of terms, i.e., $m \to \infty$, then

$$D_n = \frac{d^{n-1}}{dx^{n-1}}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^n} = \left(\frac{\psi}{\Omega}\right)^n; \qquad x < 1; \qquad n \geqq 1. \tag{76}$$

With these values of $(D_n)_m$ and an infinite number of terms, (73) is identical with the expression (37) obtained by Hallén. Furthermore, if $a \to 0$, $\psi \to \Omega$ for all values of βh, so that (73) approaches (37) as the radius a approaches zero.

It is important to note that if a finite number of terms is used in (73), all terms belonging to a given order m of solution must be retained and no others. That is, if

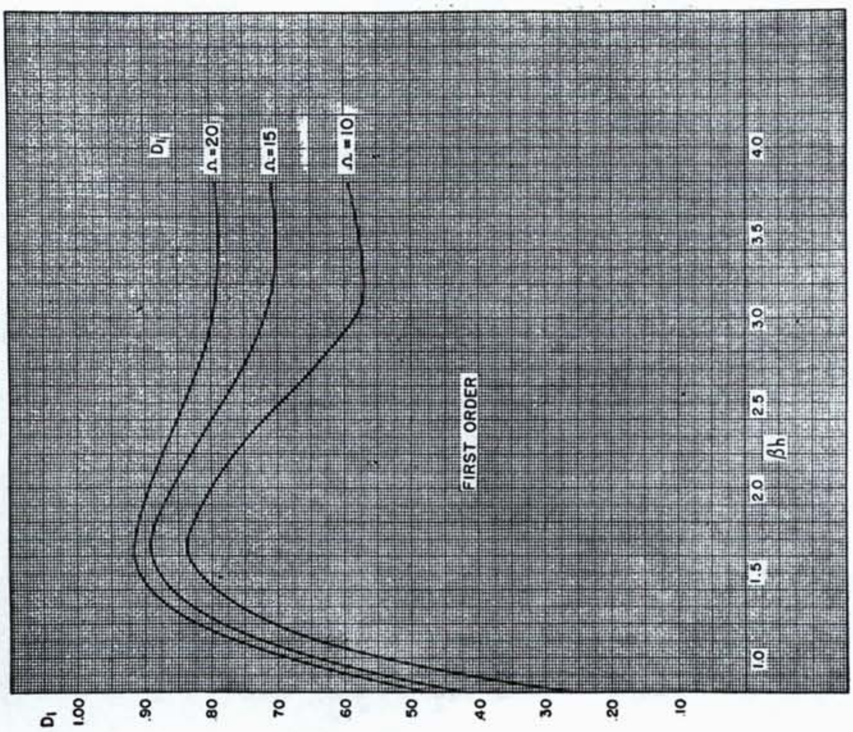

FIG. 13. The quantity D_1 for the first order theory as a function of βh, $\Omega=10, 15, 20$.

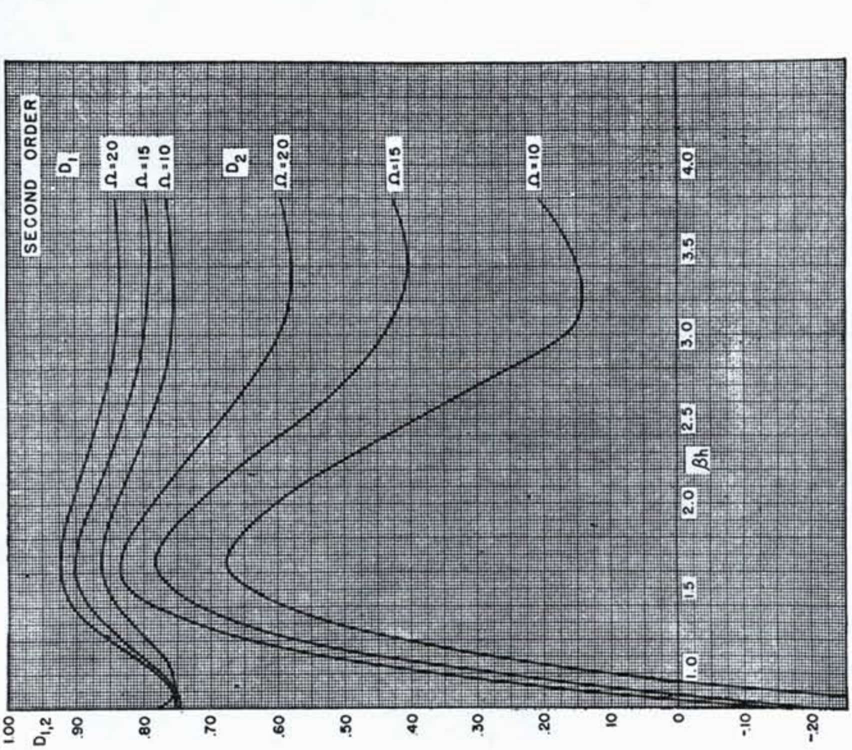

FIG. 12. The quantities D_1 and D_2 for the second order theory as a function of βh, $\Omega=10, 15, 20$.

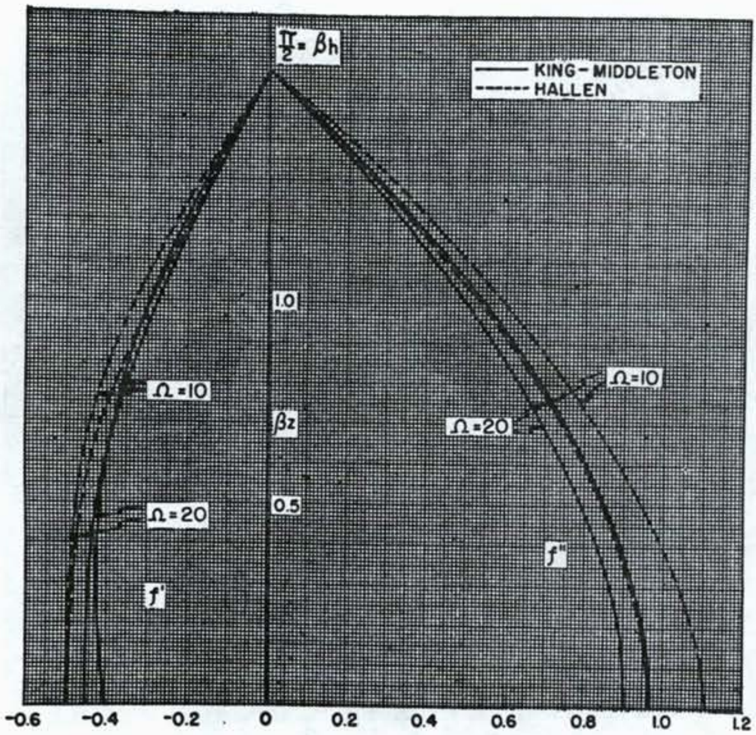

FIG. 14. First order current for $\beta h=\pi/2$ in units of $V_0^*/60\,\Omega D_H$.

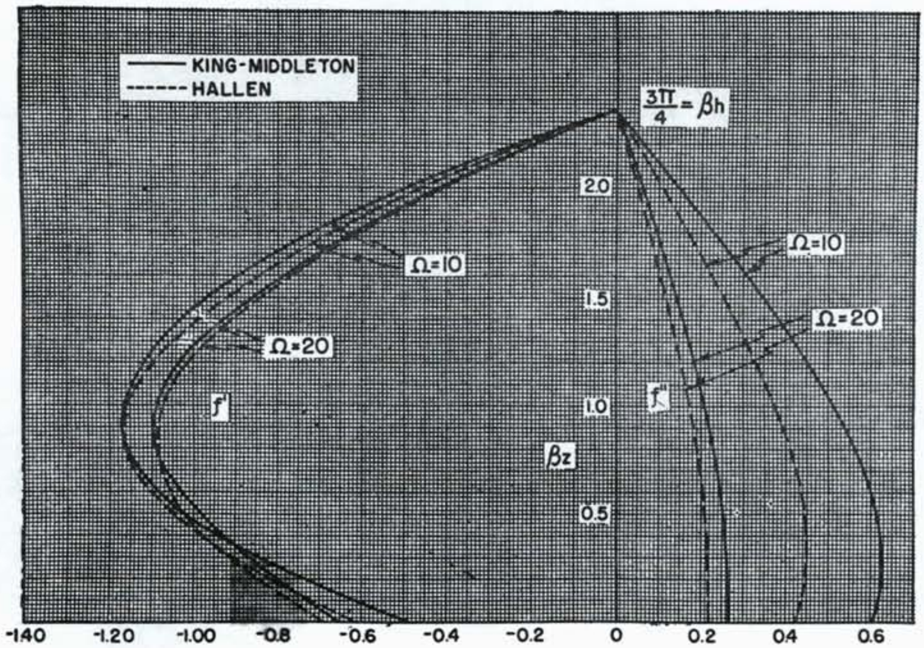

FIG. 15. First order current for $\beta h=3\pi/4$ in units of $V_0^*/60\,\Omega D_H$.

an mth order solution is evaluated, only terms contributed by $M_{nK}(z)$ and $F_{nK}(h)$ with $n=0, 1, 2, \cdots, m$ are used. It is readily verified that this is equivalent to writing

$$
\begin{aligned}
(D_1)_2 &= 1 + \left(1 - \frac{\Omega}{\psi}\right) + \left(1 - \frac{\Omega}{\psi}\right)^2, \\
(D_2)_2 &= 1 + 2\left(1 - \frac{\Omega}{\psi}\right), \\
(D_3)_2 &= 1.
\end{aligned}
\tag{77a}
$$

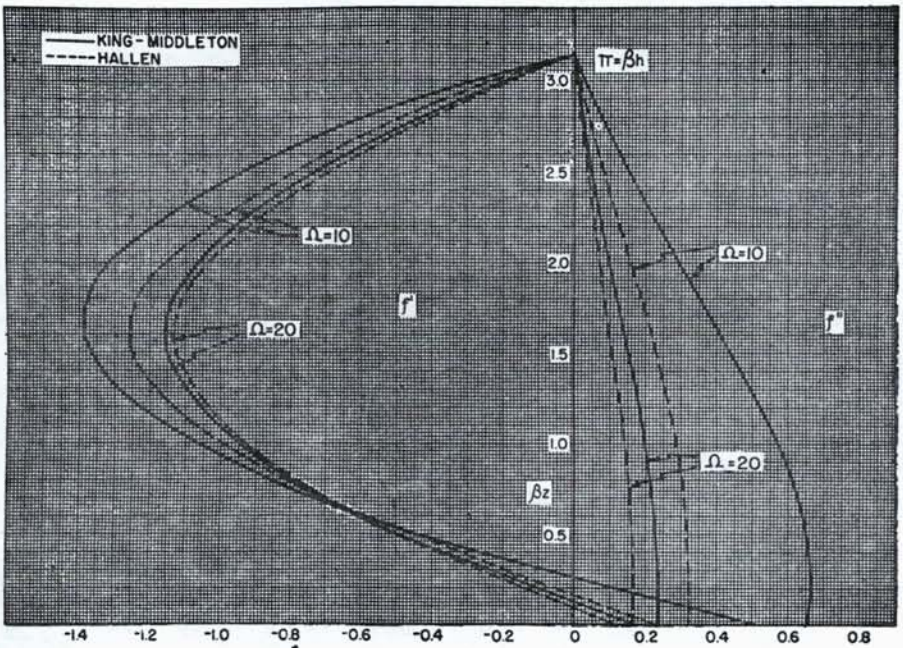

Fig. 16. First order current for $\beta h=\pi$ in units of $V_0^e/60\Omega D_H$.

The expressions $(D_1)_2$ and $(D_2)_2$ are plotted as functions of βh for $\Omega=10, 15, 20$, in Fig. 12. Similarly it is correct to set

$$
\begin{aligned}
(D_1)_0 &\equiv 1, \\
(D_1)_1 &= 1 + \left(1 - \frac{\Omega}{4}\right), \\
(D_2)_1 &= 1.
\end{aligned}
\tag{77b}
$$

The function $(D_1)_1$ is shown in Fig. 13 for $\Omega=10, 15, 20$.

The *first order* distribution of current as calculated on the one hand from the Hallén formula (38) in reference 3, and on the other hand, from (73), are shown in Figs. 14–16 for $\beta h=\pi/2, 3\pi/4, \pi$, and with $\Omega=10, 20$. The function f'' and f' when (73) is written in the form

$$
(I_z)_1 = \frac{2\pi V_0^e}{R_c\Omega D_H}(f'' + jf') \tag{78}
$$

are plotted in the figures. Numerical values of D_H are given in reference 3 where D is written instead of D_H.

Apart from the change in the input current which is discussed below in terms of the impedance, the general shape of the two sets of curves is much the same. The new, more exact theory leads to a distribution with somewhat greater relative amplitudes nearer the outer parts of the antenna, and with a somewhat larger component in phase with the driving potential difference. For $\beta h=\pi$ the *first order* solution of the new theory is the same as the first order solution of Hallén's theory if ψ is written for Ω. Since ψ is less than Ω, this means the new first order distribution is the same as Hallén's first order distribution for an antenna of greater radius, but only for $\beta h=\pi$.

6. The impedance. The formula for the impedance according to the new, more exact theory is

$$(Z_0)_m = \frac{-jR_c}{2\pi}\left\{\frac{\cos\beta h + \sum_{n=1}^{m}(D_n)_{m-1}\alpha_n/\psi^n}{(D_1)_m\sin\beta h + \sum_{n=1}^{m}(D_{n+1})_m\beta_n/\psi^n}\right\} \tag{79}$$

where α_n and β_n are defined in (38b, c); α_1 is tabulated in reference 11; α_2 in Table I. Curves for $(R_0)_m$ and $(X_0)_m$ as calculated from (79) are given in Figs. 17–19. Both second and first order solutions are shown for $\Omega=10$, 15, 20. These are calculated from

$$(Z_0)_m = 60\psi\left|\frac{A_1+jA_2}{B_1+jB_2}\right|e^{j(\tan^{-1}A_2/A_1-\tan^{-1}B_2/B_1)}, \tag{80}$$

where for the second order solution

$$\begin{aligned}
A_1 &= (D_1)_1\overset{II}{\alpha_1}/\psi + (D_2)_1\overset{II}{\alpha_2}/\psi^2,\\
A_2 &= -\left[\cos\beta h + (D_1)_1\overset{I}{\alpha_1}/\psi + (D_2)_1\overset{I}{\alpha_2}/\psi^2\right],\\
B_1 &= (D_1)_2\sin\beta h + (D_2)_2\overset{I}{\beta_1}/\psi + \overset{I}{\beta_2}/\psi^2,\\
B_2 &= (D_2)_2\overset{II}{\beta_1}/\psi + \overset{II}{\beta_2}/\psi^2,
\end{aligned} \tag{81}$$

with $(D_1)_2$ and $(D_2)_2$ given by (77a) and $(D_1)_1$, $(D_2)_1$ by (77b).

For the first order solution

$$\begin{aligned}
A_1 &= (D_1)_0\overset{II}{\alpha_1},\\
A_2 &= -\left[\psi\cos\beta h - (D_1)_0\overset{I}{\alpha_1}\right],\\
B_1 &= \psi(D_1)_1\sin\beta h + \overset{I}{\beta_1},\\
B_2 &= \overset{II}{\beta_1},
\end{aligned} \tag{82}$$

where $(D_1)_0$ and $(D_1)_1$ are given by (77b).

The impedance calculated according to the new, more exact theory differ considerably in some details but not in major outline from that obtained from the Hallén theory as calculated by King and Blake,[11] King and Harrison,[7] and Bouwkamp.[13] In general, resistances at antiresonance are smaller and occur at smaller values of βh; resistances at resonances are greater and likewise occur at smaller

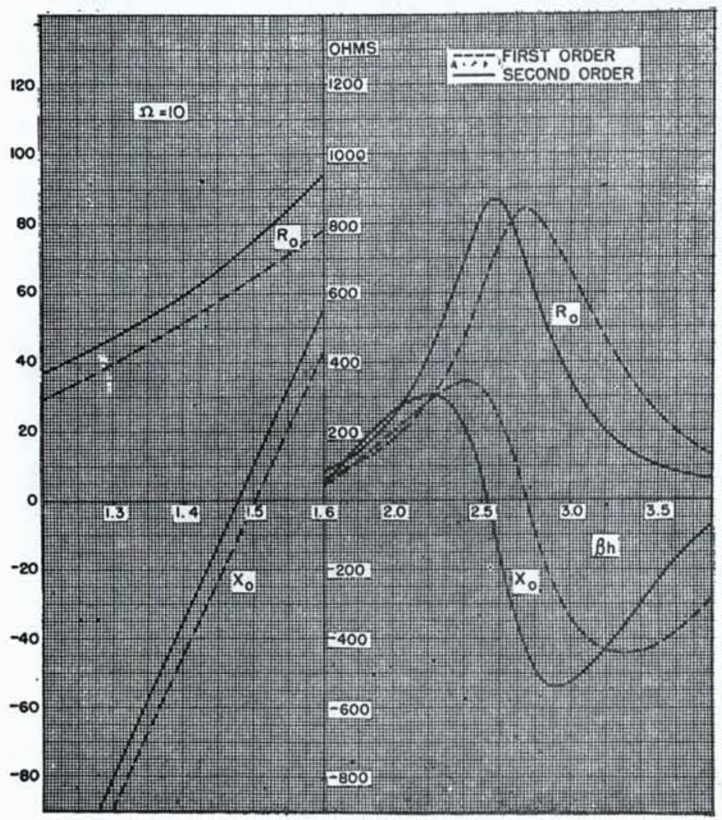

FIG. 17. The input resistance and reactance of a moderately thin, cylindrical, center-driven antenna, $\Omega=10$, or $h/a=75$, for the first and second order theories.

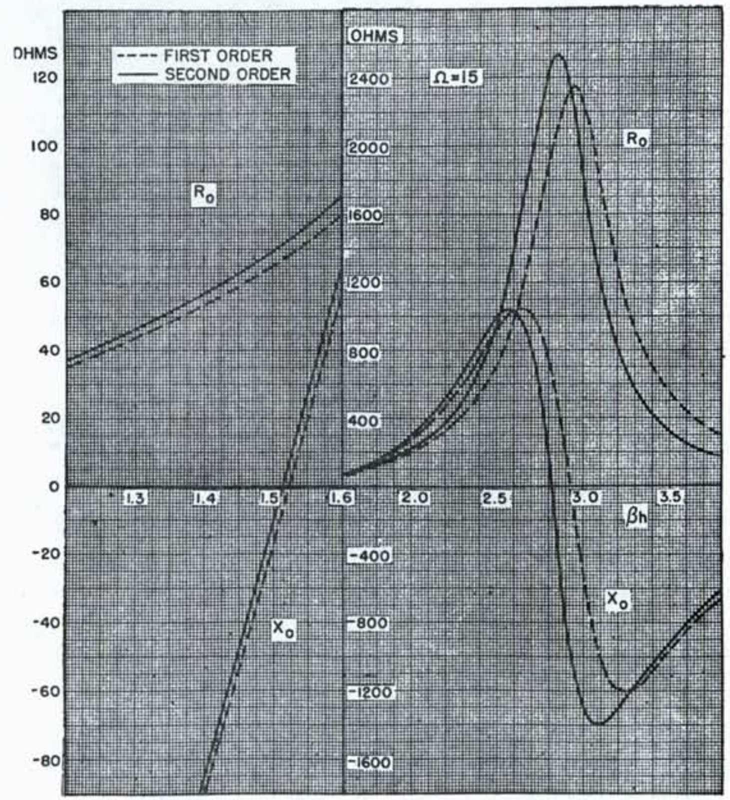

FIG. 18. The input resistance and reactance of a thin, cylindrical, center-driven antenna, $\Omega=15$, or $h/a=9.0\times 10^{2}$, for the first and second order theories.

values of βh. These differences are most significant for large values of the radius of the antenna. A critical discussion of impedance calculated from the theory here presented and comparison with experiment and with the theories of Hallén,[2,3] Gray,[12] Schelkunoff,[6,1] and others is reserved for a sequel to this paper. Accuracy of the results and convergence of the series involved also will be discussed therein.

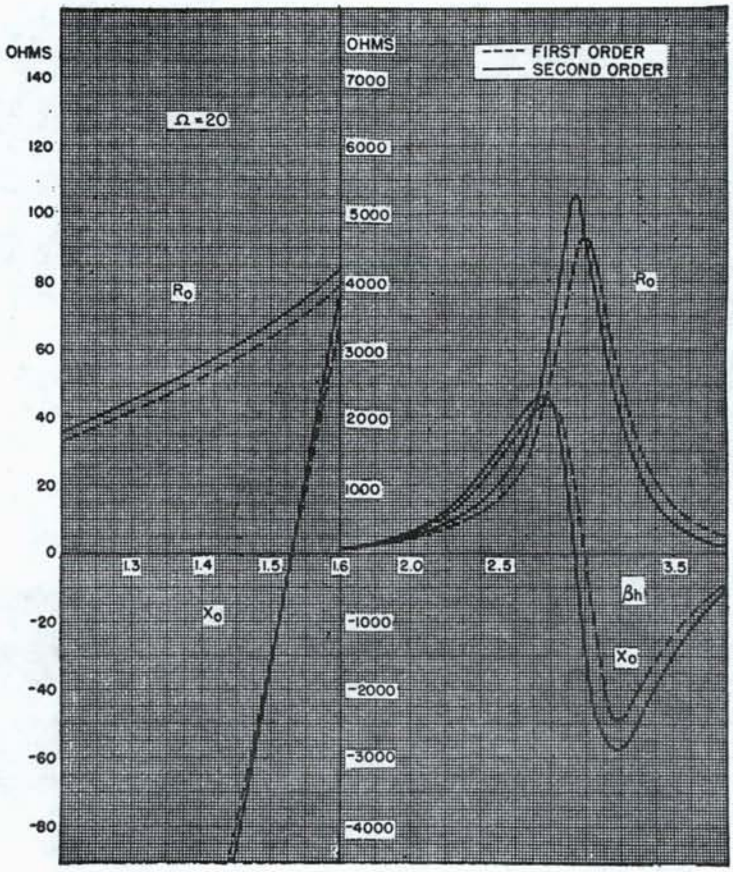

Fig. 19. The input resistance and reactance of a very thin, cylindrical, center-driven antenna, $\Omega=20$, $h/a=1.1\times10^4$, for the first and second order theories.

Using a specially constructed coaxial line and driving conditions that approximate as closely as possible the idealized slice generator D. D. King has measured the impedance of cylindrical antennas with hemispherical ends. A complete description of the apparatus, of the method, and of the results will be contained in a doctoral dissertation and in a paper to be published in another journal. A cross-section of the results involving all of the critical values for $\Omega=10$ are shown below together with the corresponding theoretical results of the theory outlined above. The agreement is seen to be good for *all* quantities in the second order theory, only approximate in the first order theory.

TABLE II

	Anti-resonant R_0	$\frac{\lvert X\rvert_{min}}{\lvert X\rvert_{max}}$	$\pi-\beta h_{anti-res.}$	Resonant R_0	$\frac{\pi}{2}\beta h_{res.}$	R_0 at $\beta h=\frac{\pi}{2}$	X_0 at $\beta h=\frac{\pi}{2}$
Experimental Results by D. D. King	800	1.95	.60	71.5	.098	85	47
Theoretical Results Second Order	860	1.80	.61	71.0	.094	88	42.5
Theoretical Results First Order	840	1.27	.39	64.8	.065	67	30

APPENDIX: INTEGRAL FUNCTIONS

The Functions $C(z)$ and $S(z)$.

$$C(z) \equiv \int_{-h}^{h} R_1^{-1}e^{-i\beta R_1}\cos\beta z'dz' = \int_0^h (R_1^{-1}e^{-i\beta R_1} + R_2^{-1}e^{-i\beta R_2})\cos\beta z'dz', \tag{1}$$

$$S(z) \equiv \int_{-h}^{h} R_1^{-1}e^{-i\beta R_1}\sin\beta\,|z'|\,dz' = \int_0^h (R_1^{-1}e^{-i\beta R_1} + R_2^{-1}e^{-i\beta R_2})\sin\beta z'dz', \tag{2}$$

where

$$R_1 = \sqrt{(z-z')^2+a^2} \quad (3a) \qquad R_2 = \sqrt{(z+z')^2+a^2}. \quad (3b)$$

These integrals can be written in the form

$$C(z) = \tfrac{1}{2}[I_1+I_2+I_3+I_4], \tag{4}$$

$$S(z) = \frac{-j}{2}[I_1-I_2+I_3-I_4] = \frac{j}{2}[I_4-I_3+I_2-I_1], \tag{5}$$

where

$$I_1 = \int_0^h R_1^{-1}e^{-i\beta(R_1-z')}dz' = e^{i\beta z}\int_0^h R_1^{-1}e^{-i\beta(R_1+z-z')}dz', \tag{6}$$

$$I_2 = \int_0^h R_1^{-1}e^{-i\beta(R_1+z')}dz' = e^{-i\beta z}\int_0^h R_1^{-1}e^{-i\beta(R_1-z+z')}dz', \tag{7}$$

$$I_3 = \int_0^h R_2^{-1}e^{-i\beta(R_2-z')}dz' = e^{-i\beta z}\int_0^h R_2^{-1}e^{-i\beta(R_2-z-z')}dz', \tag{8}$$

$$I_4 = \int_0^h R_2^{-1}e^{-i\beta(R_2+z')}dz' = e^{i\beta z}\int_0^h R_2^{-1}e^{-i\beta(R_2+z+z')}dz'. \tag{9}$$

The four integrals (6)–(9) can all be reduced to the form

$$\int_v^{v_h} v^{-1}e^{-v}dv \tag{10}$$

by making the changes in the variable and in the upper and lower limits listed in Table III.

TABLE III

Integral	v	$\frac{dv}{dz'}$	v_0 for $z'=0$	v_h for $z'=h$
I_1	$j\beta(R_1+z-z')=j\beta[\sqrt{(z'-z)^2+a^2}-(z'-z)]$	$j\beta\left[\frac{z'-z}{R_1}-1\right]=-\frac{v}{R_1}$	$j\beta(R_0+z)$	$j\beta(R_{1h}-u_1)$
I_2	$j\beta(R_1-z+z')=j\beta[\sqrt{(z'-z)^2+a^2}+(z'-z)]$	$j\beta\left[\frac{z'-z}{R_2}+1\right]=\frac{v}{R_1}$	$j\beta(R_0-z)$	$j\beta(R_{1h}+u_1)$
I_3	$j\beta(R_2-z-z')=j\beta[\sqrt{(z'+z)^2+a^2}-(z'+z)]$	$j\beta\left[\frac{z'+z}{R_2}-1\right]=-\frac{v}{R_2}$	$j\beta(R_0-z)$	$j\beta(R_{2h}-u_2)$
I_4	$j\beta(R_2+z+z')=j\beta[\sqrt{(z'+z)^2+a^2}+(z'+z)]$	$j\beta\left[\frac{z'+z}{R_2}+1\right]=\frac{v}{R_2}$	$j\beta(R_0+z)$	$j\beta(R_{2h}+u_2)$

$R_0=\sqrt{z^2+a^2}$; $u_2=h+z$; $u_1=h-z$; $R_{2h}=\sqrt{u_2^2+a^2}$; $R_{1h}=\sqrt{u_1^2+a^2}$

In terms of exponential, sine, and cosine integrals,

$$\int_{ja}^{jb}\frac{e^{-u}}{u}\,du = \mathrm{Ei}\,(-jb) - \mathrm{Ei}\,(-ja) = \mathrm{Ci}\,(b) - \mathrm{Ci}\,(a) - j\,\mathrm{Si}\,(b) + j\,\mathrm{Si}\,(a). \tag{11}$$

With (11) the several integrals (6)–(9) may be expressed as follows in terms of the exponential integral and the sine and cosine integrals

$$I_1 = -e^{j\beta z}\{\mathrm{Ei}\,[-j\beta(R_{1h}-u_1)] - \mathrm{Ei}\,[-j\beta(R_0+z)]\} \tag{12a}$$
$$= -e^{j\beta z}\{\mathrm{Ci}\,\beta(R_{1h}-u_1) - \mathrm{Ci}\,\beta(R_0+z) - j\,\mathrm{Si}\,\beta(R_{1h}-u) + j\,\mathrm{Si}\,\beta(R_0+z)\}. \tag{12b}$$

$$I_2 = e^{-j\beta z}\{\mathrm{Ei}\,[-j\beta(R_{1h}+u_1)] - \mathrm{Ei}\,[-j\beta(R_0-z)]\} \tag{13a}$$
$$= e^{-j\beta z}\{\mathrm{Ci}\,\beta(R_{1h}+u_1) - \mathrm{Ci}\,\beta(R_0-z) - j\,\mathrm{Si}\,\beta(R_{1h}+u_1) + j\,\mathrm{Si}\,\beta(R_0-z)\}. \tag{13b}$$

$$I_3 = e^{-j\beta z}\{\mathrm{Ei}\,[-j\beta(R_{2h}-u_2)] - \mathrm{Ei}\,[-j\beta(R_0-z)]\} \tag{14a}$$
$$= e^{-j\beta z}\{\mathrm{Ci}\,\beta(R_{2h}-u_2) - \mathrm{Ci}\,\beta(R_0-z) - j\,\mathrm{Si}\,\beta(R_{2h}-u_2) + j\,\mathrm{Si}\,\beta(R_0-z)\}. \tag{14b}$$

$$I_4 = e^{j\beta z}\{\mathrm{Ei}\,[-j\beta(R_{2h}+u_2)] - \mathrm{Ei}\,[-j\beta(R_0+z)]\} \tag{15a}$$
$$= e^{j\beta z}\{\mathrm{Ci}\,\beta(R_{2h}+u_2) - \mathrm{Ci}\,\beta(R_0+z) - j\,\mathrm{Si}\,\beta(R_{2h}+u_2) + j\,\mathrm{Si}\,\beta(R_0+z)\}. \tag{15b}$$

Upon combining the several integrals according to (4) and (5),

$$C(z) = \tfrac{1}{2}e^{j\beta z}\{\mathrm{Ei}\,[-j\beta(R_{2h}+u_2)] - \mathrm{Ei}\,[-j\beta(R_{1h}-u_1)]\} + \tfrac{1}{2}e^{-j\beta z}\{\mathrm{Ei}\,[-j\beta(R_{1h}+u_1)] - \mathrm{Ei}\,[-j\beta(R_{2h}-u_2)]\}. \tag{16a}$$

$$C(z) = \tfrac{1}{2}e^{j\beta z}\{\mathrm{Ci}\,\beta(R_{2h}+u_2) - \mathrm{Ci}\,\beta(R_{1h}-u_1) - j\,\mathrm{Si}\,\beta(R_{2h}+u_2) + j\,\mathrm{Si}\,\beta(R_{1h}-u_1)\} + \tfrac{1}{2}e^{-j\beta z}\{\mathrm{Ci}\,\beta(R_{1h}+u_1) - \mathrm{Ci}\,\beta(R_{2h}-u_2) - j\,\mathrm{Si}\,\beta(R_{1h}+u_1) + j\,\mathrm{Si}\,\beta(R_{2h}-u_2)\}. \tag{16b}$$

$$S(z) = \frac{j}{2} e^{j\beta z}\{\operatorname{Ei}[-j\beta(R_{2h}+u_2)] + \operatorname{Ei}[-j\beta(R_{1h}-u_1)] - 2\operatorname{Ei}[-j\beta(R_0+z)]\}$$
$$+ \frac{j}{2} e^{-j\beta z}\{\operatorname{Ei}[j\beta(R_{1h}+u_1)] + \operatorname{Ei}[-j\beta(R_{2h}-u_2)] - 2\operatorname{Ei}[-j\beta(R_0-z)]\}. \quad (17a)$$

$$S(z) = \frac{j}{2} e^{j\beta z}[\operatorname{Ci}\beta(R_{2h}+u_2) + \operatorname{Ci}\beta(R_{1h}-u_1) - j\operatorname{Si}\beta(R_{2h}+u_2)$$
$$- j\operatorname{Si}\beta(R_{1h}-u_1) - 2\operatorname{Ci}\beta(R_0+z) + j2\operatorname{Si}\beta(R_0+z)]$$
$$+ \frac{j}{2} e^{-j\beta z}[\operatorname{Ci}\beta(R_{1h}+u_1) + \operatorname{Ci}\beta(R_{2h}-u_2) - j\operatorname{Si}\beta(R_{1h}+u_1)$$
$$- j\operatorname{Si}\beta(R_{2h}-u_2) - 2\operatorname{Ci}\beta(R_0-z) + j2\operatorname{Si}\beta(R_0-z)]. \quad (17b)$$

In trigonometric form

$$C(z) = \tfrac{1}{2}\cos\beta z[\operatorname{Ci}\beta(R_{2h}+u_2) + \operatorname{Ci}\beta(R_{1h}+u_1) - \operatorname{Ci}\beta(R_{2h}-u_2)$$
$$- \operatorname{Ci}\beta(R_{1h}-u_1) - j\operatorname{Si}\beta(R_{2h}+u_2) - j\operatorname{Si}\beta(R_{1h}+u_1) + j\operatorname{Si}\beta(R_{2h}-u_2)$$
$$+ j\operatorname{Si}\beta(R_{1h}-u_1]$$
$$+ \tfrac{1}{2}\sin\beta z[\operatorname{Si}\beta(R_{2h}+u_2) - \operatorname{Si}\beta(R_{1h}+u_1) + \operatorname{Si}\beta(R_{2h}-u_2)$$
$$- \operatorname{Si}\beta(R_{1h}-u_1) + j\operatorname{Ci}\beta(R_{2h}+u_2) - j\operatorname{Ci}\beta(R_{1h}+u_1) + j\operatorname{Ci}\beta(R_{2h}-u_2)$$
$$- j\operatorname{Ci}\beta(R_{1h}-u_1)]. \quad (18)$$

$$S(z) = \tfrac{1}{2}\cos\beta z[\operatorname{Si}\beta(R_{2h}+u_2) + \operatorname{Si}\beta(R_{1h}+u_1) + \operatorname{Si}\beta(R_{2h}-u_2)$$
$$+ \operatorname{Si}\beta(R_{1h}-u_1) - 2\operatorname{Si}\beta(R_0+z) - 2\operatorname{Si}\beta(R_0-z) + j\operatorname{Ci}\beta(R_{2h}+u_2)$$
$$+ j\operatorname{Ci}\beta(R_{1h}+u_1) + j\operatorname{Ci}\beta(R_{2h}+u_2) + j\operatorname{Ci}\beta(R_{1h}-u_1)$$
$$- j2\operatorname{Ci}\beta(R_0+z) - j2\operatorname{Ci}\beta(R_0-z)]$$
$$- \tfrac{1}{2}\sin\beta z[\operatorname{Ci}\beta(R_{2h}+u_2) - \operatorname{Ci}\beta(R_{1h}+u_1) - \operatorname{Ci}\beta(R_{2h}-u_2)$$
$$+ \operatorname{Ci}\beta(R_{1h}-u_1) - 2\operatorname{Ci}\beta(R_0+z) + 2\operatorname{Ci}\beta(R_0-z)$$
$$- j\operatorname{Si}\beta(R_{2h}+u_2) + j\operatorname{Si}\beta(R_{1h}+u_1) + j\operatorname{Si}\beta(R_{2h}-u_2)$$
$$- j\operatorname{Si}\beta(R_{1h}-u_1) + j2\operatorname{Si}\beta(R_0+z) - j2\operatorname{Si}\beta(R_0-z)]. \quad (19)$$

With

$$R_0 = \sqrt{z^2+a^2}; \quad u_2 = h+z; \quad u_1 = h-z; \quad R_{2h} = \sqrt{u_2^2+a^2}; \quad R_{1h} = \sqrt{u_1^2+a^2}.$$

These are exact expressions for the integrals (1) and (2). They are valid for all values of the argument z and the parameters h and a.

The integral $E(z)$.

$$E(z) = \int_{-h}^{h} \frac{e^{-j\beta R_1}}{R_1}\,dz'. \quad (20)$$

Let the variable be changed by setting

$$\beta R_1 = \beta\sqrt{(z-z')^2+a^2} = \sqrt{U^2+V^2}, \quad (21)$$

where

$$U \equiv \beta(z-z'); \qquad V \equiv \beta a. \quad (22)$$

The integral then becomes

$$E(z) = \int_{\beta R_{1h}}^{\beta R_{2h}} (U^2 + V^2)^{-1/2} e^{-j(U^2+V^2)^{1/2}} dU$$
$$= \int_{\beta R_{1h}}^{\beta R_{2h}} (U^2 + V^2)^{-1/2} \cos (U^2 + V^2)^{1/2} dU$$
$$- j \int_{\beta R_{1h}}^{\beta R_{2h}} (U^2 + V^2)^{-1/2} \sin]U^2 + V^2)^{1/2} dU \tag{23}$$

with

$$R_{2h} = \sqrt{(h + z)^2 + a^2}; \qquad R_{1h} = \sqrt{(h - z)^2 + a^2}. \tag{24}$$

Let the following symbols be introduced:

$$\mathrm{Cuv}\, x = \int_0^x (U^2 + V^2)^{-1/2} \cos (U^2 + V^2)^{1/2} dU, \tag{25}$$

$$\mathrm{Suv}\, x = \int_0^x (U^2 + V^2)^{-1/2} \sin (U^2 + V^2)^{1/2} dU. \tag{26}$$

These functions satisfy the conditions $\mathrm{Cuv}\,(-x) = -\mathrm{Cuv}\, x$, $\mathrm{Suv}\,(-x) = -\mathrm{Suv}\, x$. In terms of the notation (25) and (26), the integral (20) becomes

$$E(z) = \mathrm{Cuv}\, \beta R_{2h} - \mathrm{Cuv}\, \beta R_{1h} - j\, \mathrm{Suv}\, \beta R_{2h} + j\, \mathrm{Suv}\, \beta R_{1h}. \tag{27}$$

This is an exact expression for the integral (20).

Approximate Expressions for $C(z)$, $S(z)$, and $E(z)$.

If the parameter a appearing in $R_{2h} = \sqrt{(h+z)^2+a^2}$, $R_{1h} = \sqrt{(h-z)^2+a^2}$ is small compared with h, useful approximate expressions for the integrals $C(z)$, $S(z)$, and $E(z)$ may be derived as follows. Expanding the integral cosine using

$$\overline{\mathrm{Ci}}\, x = C + \ln x - \mathrm{Ci}\, x = \int_0^x u^{-1}(1 - \cos u) du \tag{28}$$

where C is Euler's constant, one obtains

$$\mathrm{Ci}\, \beta(R_{2h} - u_2) = C + \ln \beta(R_{2h} - u_2) - \overline{\mathrm{Ci}}\, \beta(R_{2h} - u_2), \tag{29}$$

$$\mathrm{Ci}\, \beta(R_{1h} - u_1) = C + \ln \beta(R_{1h} - u_1) - \overline{\mathrm{Ci}}\, \beta(R_{1h} - u_1), \tag{30}$$

$$\mathrm{Ci}\, \beta(R_0 - z) = C + \ln \beta(R_0 - z) - \overline{\mathrm{Ci}}\, \beta(R_0 - z). \tag{31}$$

However,

$$R_{2h} - u_2 = \sqrt{(h + z)^2 + a^2} - (h + z), \tag{32}$$

$$R_{1h} - u_1 = \sqrt{(h - z)^2 + a^2} - (h - z), \tag{33}$$

$$R_0 - z = \sqrt{z^2 + a^2} - z, \tag{34}$$

are all of order of magnitude a, so that the arguments $\beta(R_{2h} - u_2)$ and $\beta(R_{1h} - u_1)$ are of magnitude βa. It follows that since the arguments of $\overline{\mathrm{Ci}}\, \beta(R_{2h} - u_2)$, $\overline{\mathrm{Ci}}\, \beta(R_{1h} - u_1)$ and $\overline{\mathrm{Ci}}\, \beta(R_0 - z)$ are small these functions may be expanded in series. The leading

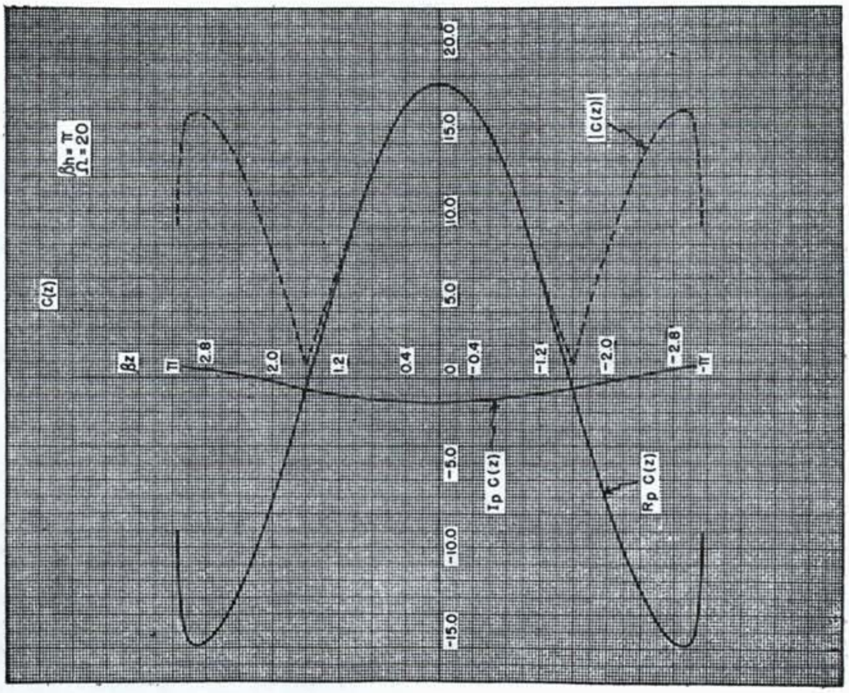

FIG. 21. The function $C(z)$ near anti-resonance, $\beta h = \pi$, $\Omega = 20$.

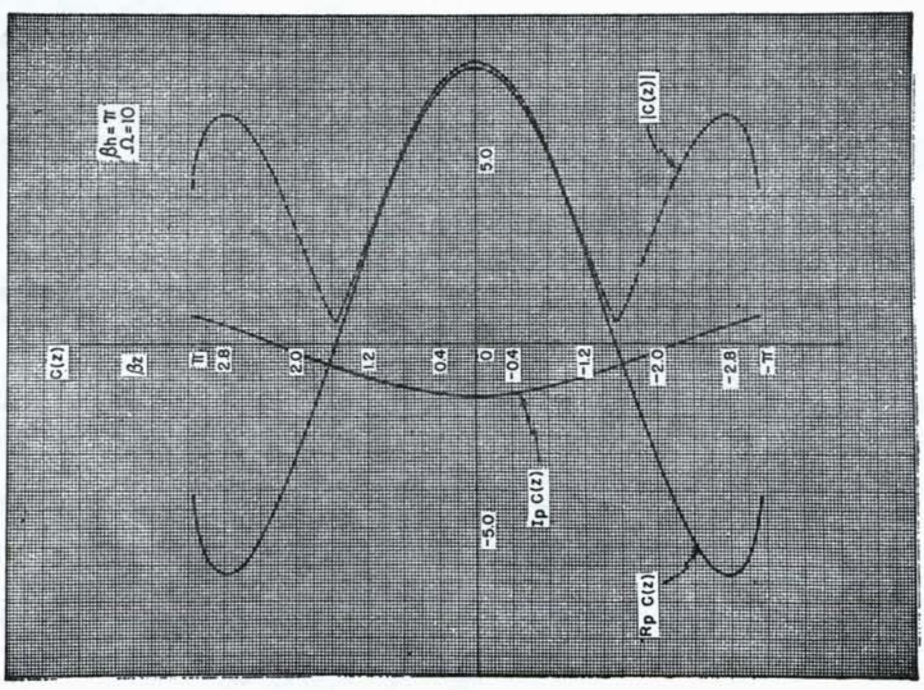

FIG. 20. The function $C(z)$ near anti-resonance, $\beta h = \pi$, $\Omega = 10$.

term in each case is of the order of magnitude $\frac{1}{4}\beta^2a^2$ so that the $\overline{\text{Ci}}$ terms in (29)–(31) are negligible compared with the logarithm. Hence,

$$\text{Ci}\,\beta(R_{2h} - u_2) \doteq C + \ln \beta(R_{2h} - u_2), \tag{35}$$

$$\text{Ci}\,\beta(R_{1h} - u_1) \doteq C + \ln \beta(R_{1h} - u_1), \tag{36}$$

$$\text{Ci}\,\beta(R_0 - z) \doteq C + \ln \beta(R_0 - z). \tag{37}$$

Since the functions Si $\beta(R_{1h}-u_1)$ and Si $\beta(R_{2h}-u_2)$ are of order of magnitude βa, they are negligible compared with Si $\beta(R_{2h}+u_2)$ and Si $\beta(R_{1h}+u_1)$ except very near the ends $z=\pm h$. If Ci $\beta(R_{2h}+u_2)$, Ci $\beta(R_{1h}+u_h)$, and Ci $\beta(R_0+z)$ are expanded using (28) and the relation

$$\ln \frac{u + (u^2 + a^2)^{1/2}}{a} = \sinh^{-1} \frac{u}{a}, \tag{38}$$

and the approximations,

$$\overline{\text{Ci}}\,\beta(R_{2h} + u_2) \doteq \overline{\text{Ci}}\,2\beta u_2 = \overline{\text{Ci}}\,2\beta(h + z), \tag{39}$$

$$\overline{\text{Ci}}\,\beta(R_{1h} + u_1) \doteq \overline{\text{Ci}}\,2\beta u_1 = \overline{\text{Ci}}\,2\beta(h - z), \tag{40}$$

$$\overline{\text{Ci}}\,\beta(R_0 + z) \doteq \overline{\text{Ci}}\,2\beta z, \tag{41}$$

(18) and (19) reduce to the following approximate forms:

$$\begin{aligned} C(z) \doteq &- \tfrac{1}{2}\cos\beta z[\overline{\text{Ci}}\,2\beta(h+z) + \overline{\text{Ci}}\,2\beta(h-z) + j\,\text{Si}\,2\beta(h+z) + j\,\text{Si}\,2\beta(h-z)] \\ &+ \tfrac{1}{2}\sin\beta z[\text{Si}\,2\beta(h+z) - \text{Si}\,2\beta(h-z) - j\,\overline{\text{Ci}}\,2\beta(h+z) + j\,\overline{\text{Ci}}\,2\beta(h-z)] \\ &+ \cos\beta z\left[\sinh^{-1}\frac{h+z}{a} + \sinh^{-1}\frac{h-z}{a}\right]. \end{aligned} \tag{42}$$

$$\begin{aligned} S(z) \doteq &\ \tfrac{1}{2}\cos\beta z[\text{Si}\,2\beta(h+z) + \text{Si}\,2\beta(h-z) - 2\,\text{Si}\,2\beta|z| \\ &- j\,\text{Ci}\,2\beta(h+z) - j\,\text{Ci}\,2\beta(h-z) + 2j\,\text{Ci}\,2\beta z] \\ &+ \tfrac{1}{2}\sin\beta z[\overline{\text{Ci}}\,2\beta(h+z) - \overline{\text{Ci}}\,2\beta(h-z) + j\,\text{Si}\,2\beta(h+z) \\ &- j\,\text{Si}\,2\beta(h-z) - 2j\,\text{Si}\,2\beta z] - \sin\beta|z|\ \overline{\text{Ci}}\,2\beta z \\ &+ \sin\beta|z|\left[\sinh^{-1}\frac{h+z}{a} + \sinh^{-1}\frac{h-z}{a}\right] \\ &- 2\sin\beta|z|\left[\sinh^{-1}\frac{h+|z|}{a} - \sinh^{-1}\frac{|z|}{a}\right]. \end{aligned} \tag{43}$$

The last factor in (43) is written in the expanded form shown in order to contain $\Psi_H(z) = \sinh^{-1}(h+z/a) + \sinh^{-1}(h-z/a)$. The remaining two terms may be written in the following approximate form if desired,

$$\begin{aligned} -\left[\sinh^{-1}\frac{h+|z|}{a} - \sinh^{-1}\frac{|z|}{a}\right] &= -\ln\frac{h+|z| + [(h+z)^2 + a^2]^{1/2}}{|z| + (z^2+a^2)^{1/2}} \\ &\doteq \ln\left(\frac{|z|}{h+|z|}\right). \end{aligned} \tag{44}$$

The function (42) is shown graphically in Figs. 20 and 21 for $\beta h = \pi$; in Figs. 4 and

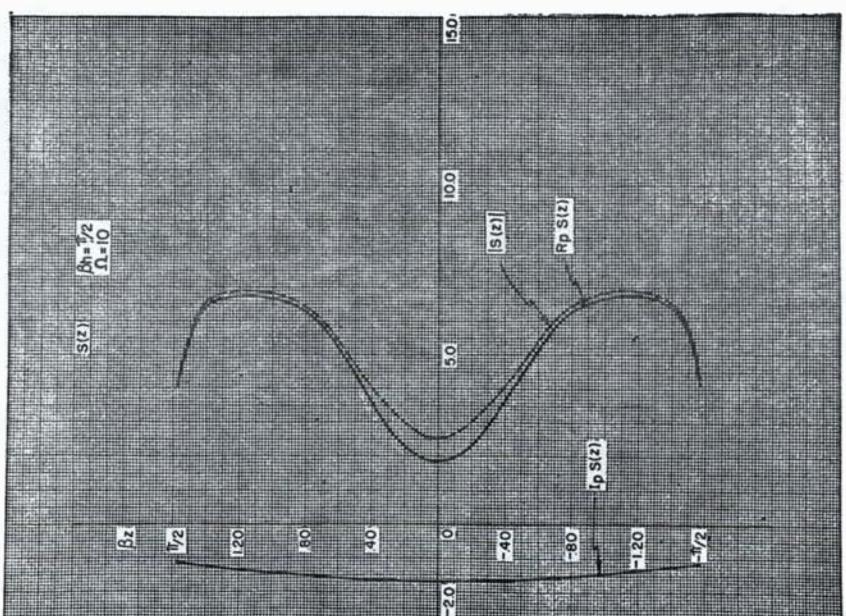

FIG. 22. The function $S(z)$ near resonance, $\beta h = \pi/2$, $\Omega = 10$.

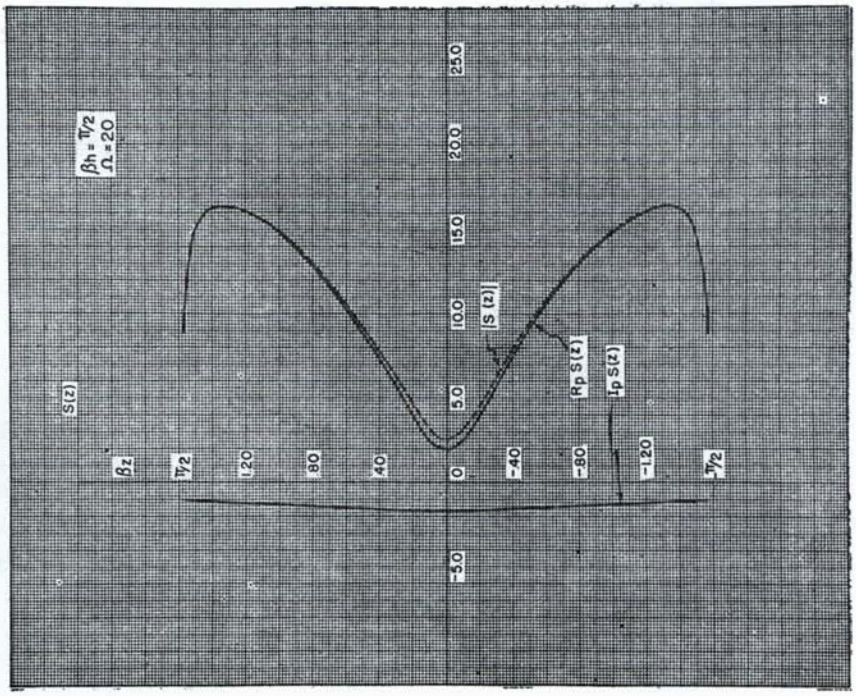

FIG. 23. The function $S(z)$ near resonance, $\beta h = \pi/2$, $\Omega = 20$.

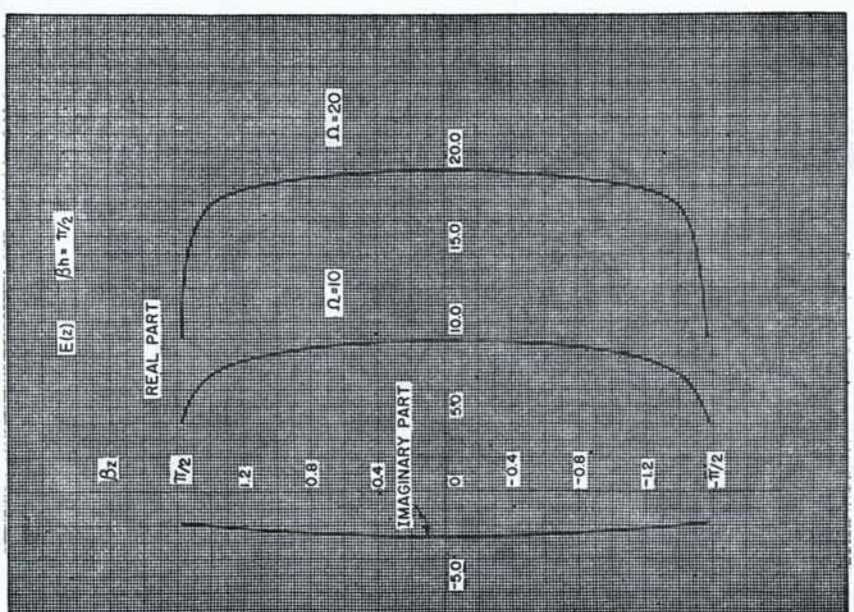

FIG. 24. The function $E(z)$, $\beta h=\pi/2$, $\Omega=10$, 20.

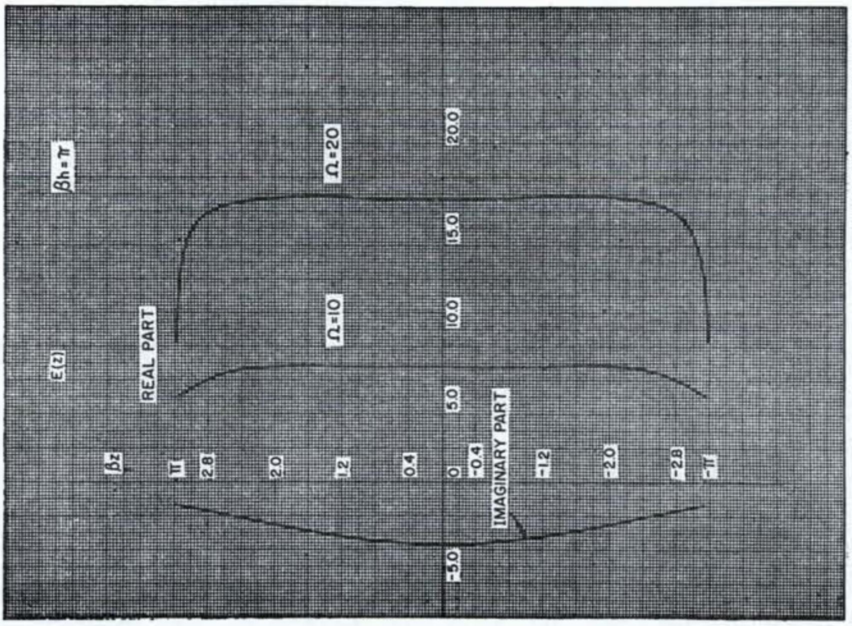

FIG. 25. The function $E(z)$, $\beta h=\pi$, $\Omega=10$, 20.

5 for $\beta h=\pi/2$. The function (43) is in Figs. 22 and 23 for $\beta h=\pi/2$; in Figs. 6 and 7 for $\beta h=\pi$.

An approximate expression for $E(z)$ is obtained from (23) by adding and subtracting

$$\int_{\beta R_{1h}}^{\beta R_{2h}} (U^2+V^2)^{-1/2} dU.$$

$$\begin{aligned} E(z) = &-\int_{\beta R_{1h}}^{\beta R_{2h}} (U^2+V^2)^{-1/2}[1-\cos (U^2+V^2)^{1/2}]dU \\ &- j\int_{\beta R_{1h}}^{\beta R_{2h}} (U^2+V^2)^{-1/2} \sin (U^2+V^2)^{1/2} dU \\ &+ \int_{\beta R_{1h}}^{\beta R_{2h}} (U^2+V^2)^{-1/2} dU. \end{aligned} \tag{45}$$

If the small quantity $V=\beta a$ is neglected in the first two integrals in (45) these remain everywhere finite and vanish at $U=0$. This is not true in the last integral in which V plays an important part. If V is neglected in the first two integrals both in the integrand and in the limits, but retained in the last integral, the following approximate expression is obtained:

$$\begin{aligned} E(z) = &-\int_{\beta(z-h)}^{\beta(z+h)} |U|^{-1}(1-\cos U)dU - j\int_{\beta(z-h)}^{\beta(z+h)} U^{-1}\sin U dU \\ &+ \int_{\beta R_{1h}}^{\beta R_{2h}} (U^2+V^2)^{1/2} dU. \end{aligned} \tag{46}$$

Because the magnitude of U and not U itself appears in the denominator of the first integral, this must be evaluated in two steps for the ranges $z'>z$ and $z'<z$. It is not necessary to write $|U|$ in the second integral because the integrand does not change sign as z' passes through z. Hence with $\overline{U}=\beta(z'-z)=-U$ the first integral in (46) becomes

$$\begin{aligned} -\int_0^{\beta(z+h)} U^{-1}(1-\cos U)dU - \int_{\beta(z-h)}^{0} \overline{U}^{-1}(1-\cos \overline{U})d\overline{U} &= -\int_0^{\beta(h+z)} U^{-1}(1-\cos U)dU \\ &\quad -\int_0^{\beta(h-z)} \overline{U}^{-1}(1-\cos \overline{U})d\overline{U}. \end{aligned} \tag{47}$$

Using (28) in (47), the sine integral in the second integral in (46), and evaluating the third integral directly, we write (46) in the form:

$$\begin{aligned} E(z) \doteq &-\overline{\mathrm{Ci}}\,\beta(h+z) - \overline{\mathrm{Ci}}\,\beta(h-z) - j\,\mathrm{Si}\,\beta(h+z) - j\,\mathrm{Si}\,\beta(h-z) \\ &+ \sinh^{-1}\left(\frac{h+z}{a}\right) + \sinh^{-1}\left(\frac{h-z}{a}\right). \end{aligned} \tag{48}$$

Use has been made of the fact that $\mathrm{Si}\,\beta(z-h)=-\mathrm{Si}\,\beta(h-z)$. The function $E(z)$ in (48) is shown graphically in Figs. 24 and 25.

The successive substitution into Eq. (21) of Eqs. (20), (1), (17), and (19) gives after simplification

$$z_1 = x_1 + iy_1 = \frac{\epsilon}{2c(1+\epsilon)^2} \tan \tfrac{1}{2}\phi + i \frac{1}{4c(1+\epsilon)^2} \tan^2 \tfrac{1}{2}\phi. \tag{22}$$

This is the parabola

$$y_1 = c\epsilon^{-2}(1+\epsilon)^2 x_1^2. \tag{23}$$

Since a change in the value of c, effects only a change of scale in the ζ-plane, c may be taken without loss of generality as

$$c = \tfrac{1}{2}\epsilon^2(1+\epsilon)^{-2}, \tag{24}$$

and this parabola becomes the one considered in Eq. (2). Setting this value of c in (19) yields

$$b = \tfrac{1}{2}(1+2\epsilon)\epsilon^{-2}. \tag{25}$$

Hence, by Eqs. (7) and (14),

$$\sigma_1^2 = (1+2\epsilon)^2/(1+\epsilon)(1+3\epsilon), \qquad \sigma_2^2 = 1/(1-\epsilon^2), \qquad g = \epsilon/(1+\epsilon). \tag{26}$$

Formulae (15) and (16) are valid for $\sigma_1 > 0$, $\sigma_2 > 0$, i.e., for $\epsilon < 1$ and thus include those profiles whose thicknesses is less than about 4/5 of their lengths.

It may also be noted that in terms of the variable γ, the slope, $\theta(\gamma)$, and the curvature, $d\theta(\gamma)/ds$, for the symmetrical profile may be written as

$$\theta(\gamma) = \gamma - \arctan \left\{ \frac{4 \tan \gamma(\tan^2 \gamma + (1+2\epsilon)/\epsilon^2)}{4 \tan^2 \gamma - [\tan^2 \gamma + (1+2\epsilon)/\epsilon^2]^2} \right\}, \tag{27}$$

$$\frac{d\theta(\gamma)}{ds} = \frac{\sec \gamma}{8c} [(1+3\epsilon)(1-\epsilon)\epsilon^{-2} \cos^4 \gamma + 6 \cos^2 \gamma - 3\epsilon^2/(1+\epsilon)^2]. \tag{28}$$

CORRECTION AND SUPPLEMENT TO OUR PAPER

THE CYLINDRICAL ANTENNA: CURRENT AND IMPEDANCE*

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BY RONOLD KING AND DAVID MIDDLETON (*Harvard University*)

Equation (58) should be written as follows:

$$\psi \equiv \overline{\Psi} = \begin{cases} |\Psi_{K1}(0)| = |\psi_1(0)|/\sin \beta h; & \beta h \leqq \pi/2 \\ |\Psi_{K1}(h - \lambda/4)| = |\psi_1(h-\lambda/4)|; & \beta h \geqq \pi/2. \end{cases} \tag{58}$$

Two lines before this equation $|\psi_1(0)|/\sin \beta h$ should be written instead of $|\psi_1(0)|$.

These changes involve no alternations in the figures. However, the function $|\psi(0)|$ plotted in Fig. 11 to the left of $\beta h = \pi/2$ is not the parameter of expansion ψ defined by (58) as modified above and as indicated in the caption. The parameter of expansion ψ as defined in (58) is plotted in Fig. 11a where the part to the right of $\beta h = \pi/2$ is the same as in Fig. 11, the part to the left of $\beta h = \pi/2$ is obtained from the curves in Fig. 11 by dividing by $\sin \beta h$.

* Received Jan. 25, 1946.

For small values of βh a convenient approximate formula is

$$\Psi_{K1}(0) = \Omega - 2 - j\beta h; \qquad \beta h < 0.5$$

so that

$$|\Psi_{K1}(0)| = \sqrt{(\Omega - 2)^2 + \beta^2 h^2} \doteq \Omega - 2 + \tfrac{1}{2}\beta^2 h^2/(\Omega - 2).$$

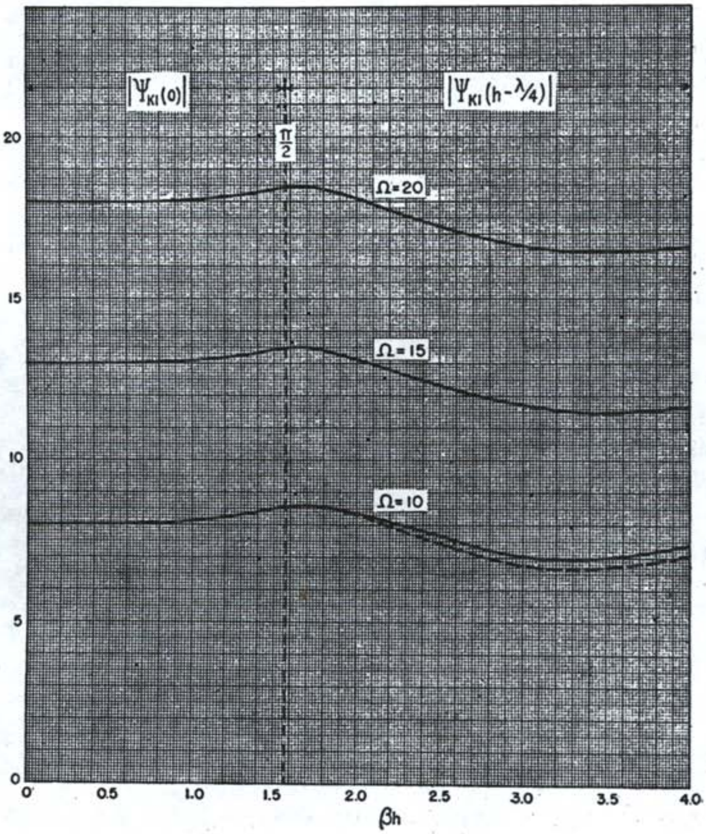

FIG. 11a. The expansion parameter ψ as defined in the corrected equation (58).

The following minor errors and misprints have been called to our attention:
page 312, Eq. (43) change Ψ to ψ,
page 319, Eqs. (59) and 62), change $\Psi_{K1}(z)$ to ψ; line following Eq. (61), delete the following: $\gamma(z) = 0$ and
page 320, Eqs. (69) and (70), change b to Ω; Eq. (76), insert $1/(n-1)!$ after the first equality sign,
page 323, Eq. (77b), change 4 to ψ,
page 324, Eq. (79), insert ψ after R_c,
page 329, Eq. (19), third line, change $(R_{2h}+u_2)$ to $(R_{2h}-u_2)$,
page 330, Eqs. (23) and (27), page 335 Eqs. (45) and (46), and in the integral preceding Eq. (45), change R_{2h} to u_2, R_{1h} to u_1, throughout,
page 330, Eq. (24) replace by: $u_2 = (h+z)$; $u_1 = (h-z)$,
page 332, Eq. (43), add superscript bar over first three symbols Ci.